IES Pons d'Icart
MATEMÀTIQUES – 2N BAT AMPLIACIÓ
Equacions i complexos II 11/XII/2003

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1. Sense utilitzar la calculadora, trobeu els valors de

a) $\binom{23}{1}-\binom{23}{3}+\binom{23}{5}-\binom{23}{7}+\cdots-\binom{23}{23}$

b) $\binom{1003}{1001}$

c) $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\cdots$

d) i^{9999} (treballant en forma polar)

e) $\dfrac{3+i}{3+4i}$

f) $\sqrt{21-20i}$

g) $(1-i)^{12}$

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a) Relacionarem aquesta suma amb el desenvolupament de $(1+i)^{23}$.

$$\binom{23}{1}-\binom{23}{3}+\binom{23}{5}-\binom{23}{7}+\cdots-\binom{23}{23}=Im\left[(1+i)^{23}\right]=Im\left[(\sqrt{2})_{\frac{\pi}{4}}\right]^{23}=$$

$$=\;Im\left[2^{\frac{23}{2}}\left(\cos\frac{23\pi}{4}+i\sin\frac{23\pi}{4}\right)\right]=Im\left[2^{11}\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)\right]=$$

$$=\;2^{11}\sqrt{2}\left(-\frac{\sqrt{2}}{2}\right)=-2^{11}=\boxed{-2048}.$$

b) $$\binom{1003}{1001}=\binom{1003}{1003-1001}=\binom{1003}{2}=\frac{1003\cdot 1002}{2}=\boxed{502503}.$$

c) Es tracta de la suma dels infinits termes d'una progressió geomètrica de raó $-\dfrac{1}{3}$.

$$1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\cdots=\frac{1}{1+\frac{1}{3}}=\boxed{\frac{3}{4}}.$$

d) $i^{9999}=(1_{90^\circ})^{9999}=1_{90^\circ\cdot 9999}=1_{360^\circ\cdot 2499+270^\circ}=1_{270^\circ}=\boxed{-i}$.

e) $$\frac{3+i}{3+4i}\cdot 3-4i3-4i=\frac{(9+4)+(3-12)i}{9+16}=\frac{13-9i}{25}=\boxed{\frac{13}{25}-\frac{9}{25}\,i}.$$

f) Utilitzarem la forma binòmica.

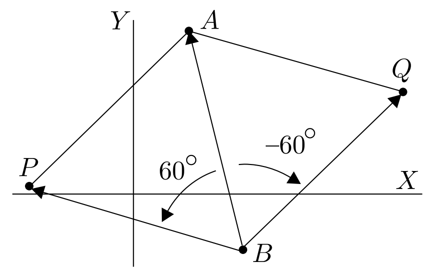
$$\begin{aligned}
\sqrt{21-20i}=a+bi &\iff (a+bi)^2=21-20i\iff(a^2-b^2)+(2ab)i=21-20i\iff\\
&\iff \begin{cases}a^2-b^2=21\\ 2ab=-20\end{cases}\iff a^2-\frac{100}{a^2}=21\quad\text{i}\quad b=\frac{-10}{a}\iff\\
&\iff a^4-21a^2-100=0\quad\text{i}\quad b=\frac{-10}{a}\iff\\
&\iff a^2=\frac{21\pm\sqrt{441+400}}{2}=\frac{21\pm 29}{2}=\begin{array}{l}\nearrow\ 25\\ \searrow\ -4\end{array}\ \text{i}\ b=\frac{-10}{a}\Longrightarrow\\
&\Longrightarrow a=\pm 5\quad\text{i}\quad b=\mp 2\Longrightarrow\sqrt{21-20i}=\begin{array}{l}\nearrow\ 5-2i\\ \searrow\ -5+2i\,.\end{array}
\end{aligned}$$

g) $$(1-i)^{12}=\left[\left(\sqrt{2}\right)_{45^\circ}\right]^{12}=\left(2^6\right)_{-540^\circ}=\left(2^6\right)_{-180^\circ}=64(-1+0i)=\boxed{-64}.$$

2. Un triangle equilàter té dos vèrtexs en els punts $A(1,3)$ i $B(2,-1)$. Trobeu els dos punts que poden ser el seu tercer vèrtex utilitzant l'àlgebra dels nombres complexos.

Anomenem P i Q els dos possibles vèrtexs. Llavors,

$$\begin{aligned}
\overrightarrow{BA} = (1,3)-(2,-1) = (-1,4) \Longrightarrow P = B + \overrightarrow{BP} &= B + \mathrm{Gir}_{60^\circ}\left(\overrightarrow{BA}\right) = \\
&= (2-i)+(-1+4i)\cdot 1_{60^\circ} = \\
&= (2-i)+(-1+4i)(\cos 60^\circ + i\sin 60^\circ) = \\
&= (2-i)+(-1+4i)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\,i\right) = \\
&= (2-i)+\left[\left(-\frac{1}{2}-2\sqrt{3}\right)+\left(-\frac{\sqrt{3}}{2}+2\right)i\right] \Longrightarrow
\end{aligned}$$



$$\Longrightarrow \boxed{P = \left(\frac{3-4\sqrt{3}}{2}, \frac{2-\sqrt{3}}{2}\right)}$$

$$\begin{aligned}
Q &= B + \overrightarrow{BQ} = B + \mathrm{Gir}_{-60^\circ}\left(\overrightarrow{BA}\right) = (2-i)+(-1+4i)\cdot 1_{-60^\circ} = \\
&= (2-i)+(-1+4i)(\cos 60^\circ - i\sin 60^\circ) = (2-i)+(-1+4i)\left(\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}\,i\right) = \\
&= (2-i)+\left[\left(-\tfrac{1}{2}+2\sqrt{3}\right)+\left(\tfrac{\sqrt{3}}{2}+2\right)i\right] \Longrightarrow \boxed{Q = \left(\frac{3+4\sqrt{3}}{2}, \frac{2+\sqrt{3}}{2}\right)}
\end{aligned}$$

3. Trobeu les arrels quartes de -1 i utilitzeu-les per descompondre el polinomi x^4+1, en factors de coeficients reals.

$$\sqrt[4]{-1} = \sqrt[4]{1_{180^\circ}} = \begin{cases}
1_{45^\circ} = \dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}\,i \\
1_{135^\circ} = -\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}\,i \\
1_{225^\circ} = -\dfrac{\sqrt{2}}{2}-\dfrac{\sqrt{2}}{2}\,i \\
1_{315^\circ} = \dfrac{\sqrt{2}}{2}-\dfrac{\sqrt{2}}{2}\,i
\end{cases}$$

Per tant,

$$\begin{aligned}
&x^6+1 = \\
&= \left(x-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\,i\right)\left(x-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\,i\right)\left(x+\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\,i\right)\left(x+\frac{\sqrt{2}}{2}+\frac{1}{2}\,i\right) = \\
&= \left[\left(x-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}\right]\left[\left(x+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}\right] = \boxed{\left(x^2-\sqrt{2}\,x+1\right)\left(x^2+\sqrt{2}\,x+1\right)}.
\end{aligned}$$

4. Deduïu la fórmula de resolució, mitjançant radicals, de l'equació

$$x^3+px+q=0.$$

Cerqueu la resolució en els apunts.