QUESTIONS ABOUT MATHEMATICAL REASONING AND PROOF IN SCHOOLS

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The role of an opening plenary session at a conference like this is, I believe, to offer at least one scene or version of context, and to raise some questions which might prove fruitful as foci for our discussions. I am mindful however that such attempts are rarely successful. I am also concerned not to impose my own thoughts about the matter, but to raise questions.

BACKGROUND

As I read the context for our discussions, in 1988 the Department of Education for England and Wales brought in 'course work' as a component of assessment in secondary schools at GCSE (16+). Course work was supposed to mean mathematical investigation[1] or exploration, sometimes structured and guided, sometimes very open and undirected. At first it contributed between 50% and 100% of students' marks, but this was within a few years reduced to between 20% and 40%. It is now around 20%.

As you may imagine, many teachers were unprepared for this innovation, and so a wide range of practices grew up. At one end of the spectrum, children met all of their mathematics through work on tasks of varying degree of openness, where some tasks might take two or three weeks work. (See Boaler 1997, Watson & Ollerton 2001). At the other end, children were given some sample coursework (something to investigate), left to get on with it, then given the assessed task, and again left to get on with it. You can imagine many practices in between, and all are probably to be found. Geometry as such, and the proofs that were supposed to be part of that geometry, had already disappeared. Even at A-level (16+ examinations), concentration was and is on procedures and techniques for solving classes of problems.

When coursework was introduced it was welcomed by those who had been experimenting in that direction, but concerns were expressed that reasoning was being relegated to investigations, and that the initial emphasis was too much on getting children to draw up a table of values based on a number of cases and then to guess some sort of a formula or expression which fitted the data (Hewitt 1992), rather than using the data to appreciate the structure generating it. I was one of several people who predicted that at some point it would be necessary to revise the curriculum advice so as to engage children in justifying their formulae and their expressions. In the event, in my view, inductive generalisation has indeed predominated, while structural deduction is rarely found. Indeed, the topics used for coursework are migrating to data-handling, so that even induction is becoming rare, much less deduction. The DfEE (now DfES), through the QCA, is now examining how children can be

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[1] The term *investigation* derives from its use over several decades in the pages of *Mathematics Teaching*, with roots in the approach taken by John Wallis (1685) who referred to 'my method of investigation' which was much despised and denigrated by Pierre Fermat.

brought to develop mathematical reasoning and to encounter and engage in mathematical proof.

During this meeting there will be presentations from projects in the UK and from abroad, looking at children's appreciation of proof, so I shall not even attempt to summarise the large corpus of research. Rather I shall offer some observations and some questions which occupy me in this context, and which I suspect may require more detailed investigation if reasoning and proof are to be incorporated satisfactorily into the curriculum. The last thing we want is a return to the memorisation of proofs as formal 'poems'.

TECHNICAL TERMS

I will consider various aspects of terms such as *proof*, *reasoning, convincing, justifying,* and *axiomatising & classifying,* as well as *rigour,* leading to some questions which I hope will inform our discussions. Along the way I run into the issue of the origins of warrants and authority, and how mathematics could be seen as empowering, through demonstrating that as a citizen in a democratic society you can be your own origin of truth and certainty, within the bounds of social conventions of responsibility and justification, through reasoning which convinces yourself and others. The questions which emerge are summarised at the end, if you want to jump to them now (notice the elementary form of reasoning: *if …*!).

I have deliberately avoided the word *argument* because of its unfortunate associations with dispute (as in the frozen metaphor "argument is war", Lakoff & Johnson 1980); Nicolas Balacheff (1999) has raised this issue, and Lynne Gordon-Calvert (1996) has nicely suggested that a more appropriate word for the production of proof through reasoning might be *discussion*, along the lines of ("argument as dance" in Lakoff & Johnson *op cit*), and prefigured in Plato's Protagoras as translated in Victorian times: "let us try the mettle of one another and make proof of the truth in conversation".

PROOF

What does *proof* mean in mathematics? A range of positions can be found amongst mathematicians, from formal and formalisable proofs in the foundations of mathematics, to reasoning which convinces a community of mathematicians (Davis & Hersh 1981). Different subtleties in what is accepted as proof can be found in different mathematical domains, such as foundations, topology, algebra, and analysis, not to say computer science, applied mathematics, and statistics. Thus we could take a Wittgensteinian-Phenomenographic approach and examine the variety of ways, the dimensions-of-variation, in which the word *proof* is used in different contexts, and we could even plump for some specific position or range of positions which, for example, locate proof as a social practice to be found amongst avowed mathematicians, and as a social practice which is in need of strengthening and developing amongst proto-mathematicians (children) in schools.

I suggest that we not spend too much time debating fine points of possible interpretations of the word *proof*. Indeed, it was because of the range of meanings in use that my colleagues and I (Mason, Burton & Stacey 1982) decided not to use the word *proof* at all when describing mathematical thinking, but rather to speak in terms of a process of refinement. At first try to convince yourself. As you become more and more convinced yourself (bearing in mind Polya's advice: "do not believe your conjectures" 1962), you try to convince friends and associates, who ask questions and indicate doubts, and to whom you can respond 'in flight', by augmenting, elaborating or offering further examples to help them 'see what you are trying to say', that is 'to see what you are seeing in the way that you are seeing it'. As you become more confident, you try to express your reasoning, your way of seeing, so that it stands alone without the need for you to be present. You try to include everything a reader will need, so that sceptics who examine your reasoning later in your absence, do not need you to interpret, elaborate, or augment.

Our developmental perspective was drawn from experience of doing and publishing mathematics, and was an attempt to avoid the apparently sudden and often stupefying jump from *assertion* to *proof* which characterises the formal presentation of mathematics, and was prevalent in the geometry curriculum of previous generations. Perhaps an extreme form is found in the beautiful *Proofs from The Book*, Aigner & Ziegler (2000), which features stunningly simple proofs of complex theorems, but which come out of the blue, yet are based on years of intimate contact with examples and structures.

Support for our description comes from the origins of the word *theorem* which derives from the Greek *theorein*, meaning 'seeing'. Thus a theorem can be thought of as 'a way of seeing something', or as 'something to be seen'. A *proof* then becomes an attempt to get other people to 'see what you see'. This links with one of Bell's (1976) roles for proof as *illumination*, the others being *verification-justification* and *systematisation*. Hanna and Jahnke (1993) suggest that for a novice, understanding (illumination) is of maximum importance as a preliminary step towards appreciating what it is that is being justified. It is unlikely that children in school, perhaps even in college, experience proof as a process of systematisation which is more associated with a Euclid-Bourbaki programme of mathematics for mathematicians, and yet systematisation is inescapable when attention is given to what is assumed and what has to be proved.

Hoyles & Healy (2000) have found recently that most students (their study was with nearly 2500 children aged 14-15 in 90 schools) base their confidence in the truth for a finite number of cases (an empirical perspective of proof). I note that *demonstration* of validity by checking a few cases, has diverged from the French use of the same word. In English, it is meant, I think, to be a checking process in order to gain confidence before embarking on a proof, whereas in French it is, I believe, what in English would be called a (perhaps not always formal) proof.

Most of us probably learned to prove by being immersed in lectures, texts, and exercises in which proofs were the backbone. But we were already mathematically oriented. This level of description is too vague. Can we be more precise? This leads me to my first question:

> How did you learn what a proof was, what it means to prove something mathematically, and how to do it yourself?

WARRANTS, AND ORIGINS OF AUTHORITY

Young children are surrounded by assertions made by adults about matters which are often beyond the child's experience. It is not surprising therefore that their first *authority* is the adult. If an adult says something is the case, then it is the case, especially in matters to do with practices in the home and in other institutions such as school, shops, church, etc.. Assertion by an adult constitutes a warrant.

In interactions with children, adults also offer reasons for doing things. These vary from the structural ("we wash our hands so as to avoid getting sick from germs") to the authoritarian ("because I tell you", perhaps with the implication that it cannot be explained, or simply will not).

Many children (perhaps though, not all) go through a period of asking "why?", "why?", "why?", to the extent of exasperating the adults whom they are questioning. It is a moot point as to what children are actually seeking, as it is likely to be a mixture of exercising a newly found power to obtain adults' attention, getting a rise out of adults, exercising a power to direct conversation and hence to participate in adult-type interactions, and a desire to find out reasons.

The effect of all this socialisation is that children at school have been exposed to a variety of reason-giving situations form a variety of types of authorities. Furthermore, most children at school quickly learn that there are genuine enquiries from someone who wants to know something ("which way do I go to get to a chemist?"), social questions ("What did you do at school today?", "How are you?") which lubricate social interaction but in which the actual answer is of little interest, and teacherly enquiries which are some combination of testing, focusing of attention, and controlling behaviour (Ainley 1987, Love & Mason 1982, see also Chazan 1993).

It is not at all surprising therefore that when a teacher asks "how do you know?" or "why?", children expect the question to be a combination of social lubrication and teacherly, with at best a small element of enquiry. It is also not surprising that their answers tend to be of the form "just because" or "it just is", or "*X* said so". Thus for young children, authority is external, expressed and warranted through adults who know more, or implicit within the structure of "how things are". However, belief alone is inadequate as a warrant in mathematics (Rodd 1998). Mathematics offers an entirely different form of warrant for belief, and source of authority.

The study by Hoyles & Healy reported earlier points to a phase of being convinced by a collection of particular cases. But is this because the students do not appreciate the force of the implied generality, the range-of-change or the dimensions-of-variation (Marton & Booth 1997) encompassed? Or is it because they are immersed in a world in which they have to induce and abduce (but not deduce) from a few particulars (e.g. in locating social norms in institutions and in peer groups)?

This leads me to the questions:

> What types of tasks, what types of interactions, promote and develop a shift from "because" and "it just is" to being convinced oneself not just in the correctness of some particular cases, but through reasoning based on experience etc.? How does this occur in other subjects?
>
> Do students appreciate the generality encompassed by mathematical statements? If so, why do they act as they do; if not, what can be done to develop their awareness?

and a related question:

> Why do so many children find mathematics to be mysterious and without reason, a random collection of unjustified manipulations of symbols?

For example, it seems that children are well able to make use of counter-examples in subjects such as history or literature, where they can draw upon their own experience to provide examples (Wortham 1996). Does this happen routinely within mathematics? If so, how can it be built upon, and if not, why not?

One of the features of mathematics is that it lends itself, indeed it depends upon, a shift from external authority to the authority of one's own reasoning. However this is not an easy shift to make, for reasons which I suspect will emerge from the various contributions to this meeting.

REASONING

Despite conclusions drawn from Piaget and Inhelder's investigations (1958) that formal reasoning is a late stage of development, even very young children can use *if … then …* constructions spontaneously and correctly. I recall a young friend of mine whom I shall call Q. (aged about 3 or 4) who, sitting in his highchair at supper, suddenly announced that "if the canal rises our feet will get wet". All of this must have been in his imagination, for the canal had never risen much less flooded, though he had seen lakes at different levels at different times. What is important for me here is that he was apparently playing with possibilities in his imagination. This leads me then to a question:

> What contribution towards reasoning, and towards adopting the discourse of reasoning, is made by manipulating mental images and considering possibilities?

Reasoning is partly learning to use the discourse of *if … then …*, of making assumptions and drawing conclusions. Where do children encounter adults actually reasoning? I suspect, but perhaps others will be able to present evidence, that in few classrooms do children experience the teacher explicitly engaging in a chain of reasoning, other than when going through the step of a solution to a problem, a process which is itself problematic in other ways ('when is an example exemplary?: Mason & Pimm 1984, Anthony 1994, Rowland 1998, Bills & Rowland 1999). Posed as a question:

> In what sorts of circumstances, and in what topics do children experience a teacher (or other adult) forming a chain of reasoning? What possibilities are afforded by different topics? What sorts of structures do teachers use in such circumstances, and what are children attending to when it is going on? How might their attention be focused on the reasoning itself rather than the method or the answer?

For example, probability calculations based on spinners and dice refer to features of the apparatus (usually just counting) for justifying conclusions. Are longer chains of reasoning possible and common?

The essence of reasoning is 'because', or 'if … then …'. But in order to make sense, what is offered as the 'because' has to be more certain than what is being justified. In other words, you have to be able to put your feet on solid ground in order to engage in effective reasoning. This is what undergraduates find so difficult: knowing what you can assume and what you have to prove. What you think you know, what seems evident, sometimes has to be justified, and sometimes can be assumed. The evidence of eyes and ears, or experience, of 'just knowing', is no longer adequate. Finding firm ground upon which to stand (one of the roles of axiomatisation) itself becomes problematic.

JUSTIFYING

To *be responsible* means literally to be able to *spond* from the Latin, meaning 'to offer explanation'. Thus to become responsible is to be able to justify your actions to others. But of course if the 'others' are in authority, your justification has to be in their terms[2]. Thus justification is tied up with learning what is acceptable, what is *reasonable* (!!) as explanation. For example, it is a well known phenomenon that students entering their first course in analysis at university are often perplexed by what they are permitted to assume and take as 'true', and what they have to prove, because they are told that everything they thought they knew about numbers and functions will now be 'proved' (to varying degrees of rigour). Those who catch on soon realise that once something has been proved, they can act as before by taking it to be true. But often it is difficult to find somewhere to stand and from which to justify everything else.

This difficulty is reflected in the history of mathematics and the relatively late emergence of axioms for numbers and probability for example, and other structures such as rings, fields, groups, vector-spaces etc.. It is perhaps curious that once mathematician have struggled to identify the essence of a mathematical structure based on extensive experience of examples and of facts about those examples, they then choose to present students with the axioms followed by a long string of deductions, as if somehow students can short-cut the experience-gathering of earlier generations.

> What is involved in moving from justifying actions without being explicit about what is taken-as-accepted, to being explicit about what is assumed, leading to axiomatising or ground-establishing?
>
> Is reasoning based on numbers easier because the properties of numbers being assumed are already part of the enculturation of primary school, and is geometry harder to base soundly because once you decide to be explicit about axioms, you have a long road to get to anything surprising or interesting, or is it the other way round, or neither?

AXIOMATISING & CLASSIFYING

Algebra is a domain in which mathematicians expect that reasoning will naturally take place, indeed that it exemplifies characteristic mathematical reasoning. But when presented in school, algebra is usually experienced as 'the arithmetic of letters', as 'rules for calculating with letters'. Algebra is much more than the principal language of (mathematical) reasoning, because it is also the study of the structure of that reasoning itself, and of the structures within which that reasoning is fruitful. As such I consider it to be both within the reach of, and vital for, every active citizen. In elementary algebra, the foundations for reasoning can be fairly explicit: arithmetic is assumed, and algebra is about justifying conjectures concerning generality.

I myself do not believe that locating and working with axioms is as complicated as many students find it or as complicated as many textbooks make it. I suspect that few children have participated in the explicit unfolding and elaborating of properties of objects (*cf*. the description of level 3, van Hiele 1986), and then experienced the reverse move of taking those properties as defining, and engaged in classification of the objects which satisfy those properties. This process may sound complicated, but it is possible, even essential, if only in small measure, long before college or sixth-form. (See for example Watson & Mason 1998).

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[2] It is perhaps curious that the word *justify* contains the word *just*, given that very often children, when challenged by someone in authority to justify their actions, begin with 'I was just …".

A good example is provided by David Fielker's (1983) brilliant work on getting children to classify quadrilaterals by their diagonals as but one example of how geometry need not look like Euclid. A simple algebraic example arises from the process of writing down the properties of an even number, and then deciding which of those could be taken as definition from which the others could be deduced, leading to questions of whether zero is an even number, or –2. More complicated examples include classifying the properties of numbers which arise as a result of a particular kind of calculation such as one more than the product of four consecutive numbers, or the sum of an even number of odd numbers, or exploring the arithmetic of the numbers expressible as integer plus integer multiple of $\sqrt{5}$, as well as addressing the question 'what is a number?' by elaborating properties of numbers and thereby producing the 'rules' of algebra.

Classification is not just about making distinctions and describing properties, but about justifying conjectures that all possible objects with those properties have been described or otherwise captured.

The purpose of a definition, as Lakatos (1976) argued, is to make a theorem or collection of theorems provable. The fact that this then raises the question of what class of objects satisfy the theorem is one of the excitements of mathematics, and one of the sources of the fundamental theme of *extending meaning*.

For example:

> We want to be able to solve x + 3 = 0 and to continue to work with the usual rules of arithmetic, so we define the solution to be 'that number which when added to 3 gives 0', and then to denote it by –3. Similarly for $3x = 2$ and for $x^2 = -1$. At each stage, the meaning ascribed to the term *number* is extended.
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> The area of a trapezium, when specialised to one of the parallel edges having zero length, gives the area of a triangle correctly (a theorem), so we choose to extend meaning and allow a triangle to be a special case of a trapezium: meaning is extended to cover more cases.
>
> Many children initially reject the possibility of a non-convex quadrilateral (more generally, polygon), and even more reject ones with edges which cross. Yet they satisfy all the properties children know of such figures, but lead to complications concerning once-familiar constructs such as the 'interior angle'.
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> Functions, fundamental to the development of 20th century mathematics, are far more numerous and varied than the simple functions used in the current school curriculum. But it is precisely because the things we can say about functions actually apply to many many more complicated ones that mathematics is powerful and important.

These are just some examples of domains of fruitful experience of how it is useful to extend meaning, based on accepting a definition in terms of properties, arising from a theorem which the definition enables.

The process of property-locating, classifying, and then 'defining through requiring properties so as to make a collection of theorems provable' is actually fundamental to mathematics. Almost always, this process opens up the possibility of hitherto unexpected objects satisfying the property chosen as defining (for example, the possibility of having the numerator of a fraction larger than the denominator, or of having a function which is no-where differentiable but every-where continuous, a triangle with one angle of 180 degrees, a quadrilateral which crosses itself, and so on).

This leads me to the question:

> Would children's sense of and experience of proof be enhanced by experiencing the descriptions of properties of objects and then the reverse move of classifying all objects satisfying those properties, in a variety of different topics?

CONVINCING

Being convinced oneself is quite different from convincing someone else, particularly when that person is being sceptical. Learning to be sceptical oneself, to locate gaps and jumps in other people's reasoning is important not just in mathematics, but as an involved citizen. As my remarks have suggested repeatedly, convincing, reasoning, proving, justifying are all emergent and developing processes, not rigid formats to be learned or followed. Learning to be your own authority in mathematics, with being responsible through being able to justify, with learning how to convince yourself, a friend and a sceptic, probably reflects, or could reflect, perhaps even presage the same processes within a broader social context. Thus mathematics could speak to the experience of students at all ages.

> What is the psychological experience of moving from convincing through 'arguing' to convincing through appeal to accepted principles and facts? How might teachers be supported in supporting this transition in their students?

FOR WHOM, REASONING AND PROOF?

> Who needs to learn about mathematical reasoning and proof, why, and what reasons will be convince to stakeholders?

If it is to be only for some children, then it will contribute to the growing trend of divisiveness in the school curriculum; if for all, then it needs to be tailored so as to contribute to children's growing sense of personal responsibility and authority. Many people complain that at school they gave up on mathematics because no-one ever explained things to them (explanations may have been given of course, but may not have been recognised as such). A revised curriculum could do something about this. Experience of mathematical justification, as I have argued above, is vital for all concerned and responsible citizens, because it contributes to a personal and social shift from dependency on authority to independence of thought, from accepting external authority to questioning of those authorities. This is the essence of democratic participation, and vital if democratic structures are to prevail. Mathematics could play a lead role in the development of personal authority and reasoning which convinces yourself and others, as warrant for assertions, whether mathematical or not.

Finally,

> Is it possible to build reasoning into a National Curriculum without it turning (through the *didactic transposition*) into yet another collection of mechanical procedures and techniques to be learned?

QUESTIONS SUMMARISED

- How did you learn what a mathematical proof was, what it means to prove something mathematically, how to do it yourself?

- What types of tasks, what types of interactions, promote and develop a shift from "because" and "it just is" to being convinced oneself not just in the correctness of some particular cases, but through reasoning based on experience etc.? How does this occur in other subjects?
- Do students appreciate the generality encompassed by mathematical statements? If so, why do they act as they do; if not, what can be done to develop their awareness?
- Why do so many children find mathematics to be mysterious and without reason, a random collection of unjustified manipulations of symbols?
- What role is played in this shift by experiencing potential infinity and generality, in the sense of being able to make assertions about situations which can only be imagined, not encountered (e.g. extremely large numbers, very complex situations, things which can only be imagined not touched)?
- What contribution towards reasoning, and the discourse of reasoning is made by manipulating mental images and considering possibilities?
- In what sorts of circumstances, and in what topics do children experience a teacher forming a chain of reasoning? What possibilities are afforded by different topics? What sorts of structures do teachers use in such circumstances, and what are children attending to when it is going on? How might their attention be focused on the reasoning itself rather than the method or the answer?
- Reasoning requires somewhere to stand, some things which can be assumed or taken as being the case, on which to build chains of reasoning. Where are these foundations in school mathematics at different ages and in different topics?
- What is involved in moving from justifying actions without being explicit about what is taken-as-accepted, to being explicit about what is assumed, leading to axiomatising or ground-establishing?

  Is reasoning based on numbers easier because the properties of numbers being assumed are already part of the enculturation of primary school, and is geometry harder to base soundly because once you decide to be explicit about axioms, you have a long road to get to anything surprising or interesting, or is it the other way round, or neither?
- Would children's sense of and experience of proof be enhanced by experiencing the descriptions of properties of objects and then the reverse move of classifying all objects satisfying those properties, in a variety of different topics
- What is the psychological experience of moving from convincing through 'arguing' to convincing through appeal to accepted principles and facts? How might teachers be supported in supporting this transition in their students?
- Who needs to learn about mathematical reasoning and proof, why, and what reasons will be convince to stakeholders?
- Is it possible to build reasoning into a National Curriculum without it turning (through the *didactic transposition*) into yet another collection of mechanical procedures and techniques to be learned?

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