Sequences

Lesson 1 : Introducing sequences

We are going to look at sequences and you will realise that they are fun and interesting!

What is a sequence?

Can you find one or two ordered lists of objects, numbers, words... in everyday life? Write two examples.

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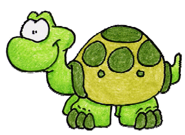
1) Read this quotation from Lewis Carroll's book 'Alice in Wonderland'



Alice: *And how many hours a day did you do lessons?*

The Mock Turtle: *Ten hours the first day, nine the next, and so on.*

Alice: *What a curious plan!*

The Gryphon*: *That's the reason they're called lessons, because they lessen from day to day.*

*A Gryphon is a legendary creature with the body of a lion and the head and wings of an eagle. Its figure is used in architecture, trademarks and modern emblems and in heraldry.

Can you write the sequence of hours the turtle has lessons?

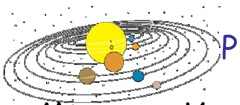
Good!! This is a **sequence of numbers** and it is easy, isn't it?

2) Are sequences an invention of mathematicians to complicate the easiest things or are they useful?

In pairs try to solve this word search and after that complete the sequences below.

A	C	K	I	R	E	N	U	J	E
Q	C	V	Z	R	N	J	U	L	Y
H	S	Y	R	E	T	I	P	U	J
U	X	I	J	B	M	N	N	L	F
N	O	V	E	M	B	E	R	I	H
E	W	R	W	E	O	P	U	R	T
Y	O	U	T	T	G	T	T	P	R
A	L	G	E	P	D	U	A	A	A
M	L	R	L	E	M	N	S	M	E
O	E	E	O	S	L	E	P	A	K
J	Y	E	I	S	U	N	A	R	U
R	V	N	V	D	L	T	I	C	A
R	R	E	B	O	T	C	O	H	X
G	I	Y	R	A	U	R	B	E	F
S	J	B	O	A	U	G	U	S	T

Earth	July	Uranus	April
Jupiter	August	Neptune	May
Saturn	September	Yellow	June
February	October	Green	Violet
March	November		



Planets

Mercury, Venus, ______ Mars, ______, ________, _______, Pluto

Calendar Months

January, ______, _______, ______, ______, _______, _______, _______, ________, _______, ________,December

Colours of the Rainbow

Red, Orange, ________, _________, Blue, Indigo, _______

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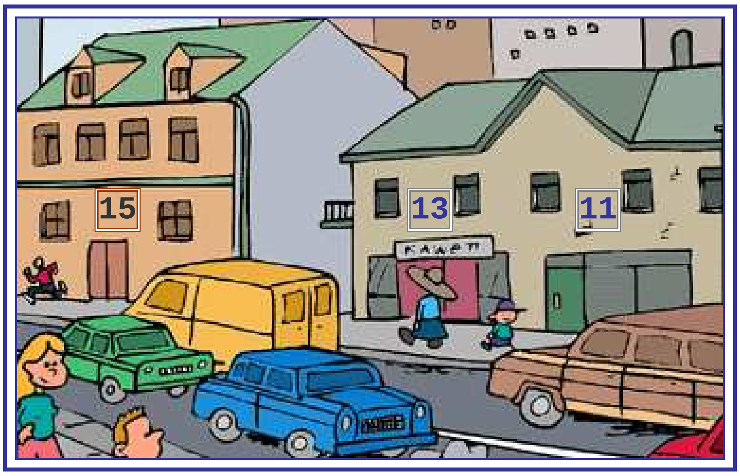
3) In pairs describe this picture.



<table>
<tr><td rowspan="4">In the</td><td colspan="4">middle</td><td rowspan="4">of the
picture</td><td rowspan="4">there is …

there are ….

I can see …</td></tr>
<tr><td>top
bottom</td><td>left
right</td><td>hand</td><td>side
corner</td></tr>
<tr><td colspan="4">background</td></tr>
<tr><td colspan="4">foreground</td></tr>
</table>

In this picture I can see…

____

____

____

- How are the houses numbered? ________________
- Write a list of numbers from houses on both sides of the street.

____, _3_,____,____,____,____,____,…

____, ____,____,_8_,____,____,____,…

Good!! These are sequences of numbers and it is easy, isn't it?

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4) Work in pairs. After doing exercise 2, you are ready to make a definition!

Use the following words.

elements.	of	sequence	an
A	list	ordered	is

Definition 1 ⇨ A sequence is_____________________________________

In this unit we are going to study sequences of numbers but, as you know, we could have a sequence of colours, musical notes, days…

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Constructing Sequences

5) You know different number sequences. Work in pairs. Match each list of numbers with a mathematical rule.

1,2,3,4,5,6,7,8,9,10…	odd numbers
2,4,6,8,10,12,14,16, …	even numbers
1,3,5,7,9,11,13,15…	powers of 2
2,4,8,16,32,64,128…	prime numbers
1,4,9,16,25,36,49…	natural numbers
1,3,5,7,11,13,17,19…	square numbers

Definition 2 ⇨ A **sequence** is a set of numbers, which are written, in some particular order. For example,

1, 3, 5, 7, 9, . . .

Here, we seem to have a rule. We have a sequence of odd numbers. It starts with the number 1, which is an odd number, and then each successive number is obtained by adding 2.

+2 +2 +2 +2 +2

1, 3, 5, 7, 9, . . .

Now write the sequence of even numbers, and its rule:

___, ___, ___, ___, ___, . . .

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Other examples are

x2 x2 x2

8, 16, 32, 64, 128,... Multiply the previous number by 2

-8 -8 -8 -8

16, 8, 0, -8, -16,... Subtract 8 from the previous number

+1 +2 +3 +4

4, 5, 7, 10, 14,... Add one extra each time

+4 +5 +9

4, 5, 9, 14, 23,... Add the previous two numbers together

6) What do you think the three dots mean?

Try to make a sentence that answers the question.

mean	sequence	they	the	goes on	that	forever

____

7) Let's define

Try to sort out these words and then, complete the definitions below with the correct word:

smtre	⇨	__terms_____	ments-le-e	⇨	_____________
red-or-de	⇨	_____________	scrip-sub-ted	⇨	_____________

Definition 3 ⇨ A **sequence** (or progression) is an ___________ list of numbers. The numbers in the list are called __________ or __________, and we represent them with a letter and a ___________ letter.

Fill in the gaps with three of these words:

term	number	counter	twelfth	index

A sequence may be named or referred to as 'A' or 'An'.

The terms of a sequence are usually named something like 'a_i' or 'a_n', with the subscripted letter '$_i$' or '$_n$' being the ___________ or ___________. So the second term of a sequence might be named 'a_2' (pronounced 'ay-sub-two'), and 'a_{12}' would designate the ____________ term.

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Complete with the correct word:

______________ A $\boxed{n}$ = 3, 9, 27, 81, 243,.... ______________

Underline the correct word(s):

If a sequence goes on forever, it is an *infinite/finite* sequence.

The numbers 1, 2, 3, 4, 5, 6, . . . n form a *finite/infinite* sequence containing just *six/n* terms. The dots indicate that *we have not written all the numbers down / the sequence goes on forever*. The n after the dots tells us that this is a *finite/infinite* sequence, and that the *last/first* number is n.

8) Write down the next three numbers in this sequence, and the rule for obtaining them using the chart below.

1,4,7,10,13,16,19,____,____,____,...

____

____

<table>
<tr><td rowspan="4">The next term in the sequence is obtained by</td><td>multiplying</td><td colspan="3" rowspan="2">the preceding term by…</td></tr>
<tr><td>dividing</td></tr>
<tr><td>adding …</td><td rowspan="2">……. one extra</td><td>… to</td><td rowspan="2">the preceding term</td></tr>
<tr><td>subtracting…</td><td>… from</td></tr>
</table>

9) Write down the rule for obtaining the next term in each of the following:

❈An= 100,96,92,88,

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The next term is obtained by subtracting four from the preceding term

❈Bn= -20,-10, 0, 10, 20…

The next term is obtained ________________________________

❈Cn= -3, -6, -12, -24…

The next term ____

❈Dn= 64, 32,16,8…

The next ____

❈En = 1,10,100,1000…

____

❈Fn=97, 96, 94, 91…

____

Are the sequences above increasing (↗) or decreasing (↘)? Fill in the chart.

Increasing sequences	Decreasing sequences

10) Look at this sequence and complete:

$A_n = 1,4,7,10,13,16,19,...$

First term ⇒ $a_1 = 1$

Second term ⇒ a__ = ___

________________ ⇒ a__ = ___

________________ ⇒ a__ = 10

________________ ⇒ a__ = 13

________________ ⇒ a__ = ___

________________ ⇒ a__ = ___

Cardinal numbers		Ordinal Numbers	
1	One	1st	First
2	Two	2nd	Second
3	Three	3rd	Third
4	Four	4th	Fourth
5	Five	5th	Fifth
6	Six	6th	Sixth
7	Seven	7th	Seventh
8	Eight	8th	Eighth
9	Nine	9th	Ninth
10	Ten	10th	Tenth
n	En	nth	enth

11) Now complete the number sequences, write the rule and what type of sequence is **increasing** (↗), **decreasing** (↘), **alternating** (↗↘) or **constant** (→).

- 3, 6, 9, 12, 15, 18, 21, 24. Rule: Add 3 ↗
- 19, 28, 37, _, _, _ Rule: ______________________________
- 1, 4, 9, 16, _, _, _ Rule: ______________________________
- -13, -25, -37, _, _, _ Rule: ______________________________
- 43, 38, 33, _, _, _ Rule: ______________________________
- 7, -7, 7, _, _, _ Rule: ______________________________
- _, _, 44, 33, 22, _ Rule: ______________________________
- _, _, -5, 0, 5, _ Rule: ______________________________
- 32, _, 32, 32, __ Rule: ______________________________
- 14, _, 28, _, 42, _ Rule: ______________________________

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Extension: Make up three number sequences for someone else to solve. One easy, one medium and one hard.

12) In activity 11 we have introduced a new concept, alternating sequences. The next sequence is an example, the terms alternating between 1 and −1,

$$1, -1, 1, -1, 1, -1, \ldots$$

- Can you find more examples?

____

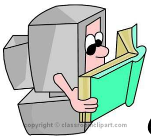
____

- Other sequences are **periodic**, and just revisit the same values over and over again. Write some examples.

0,1,2,3,0,1,2,3,0,1,2,3,...

____

____



Computers can be used for many things, even for solving mathematics!!

13) Complete the sentences below with the appropriate words:

Computers can be used for

- playing …
- writing…
- looking for…
- watching …
- listening to…
- transferring …

games | things on the Internet. | documents | files | music | TV and movies

Here is some vocabulary you will need using the Calc. Try to match each word with its definition. Look at the sheet from the Calc for help.

Active cell	Formula	Spreadsheets
Cell Reference	Graph	Value
Cell	Row	Workbook
Column	Selecting	Worksheet

____________ is a grid that organizes data.

____________ is the number that can be entered into a cell

____________ is the horizontal reference on the spreadsheet

____________ is a visual representation of data

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___________ is the cell you are currently working on

___________ is to highlight a set of cells

___________ is the vertical reference on the spreadsheet

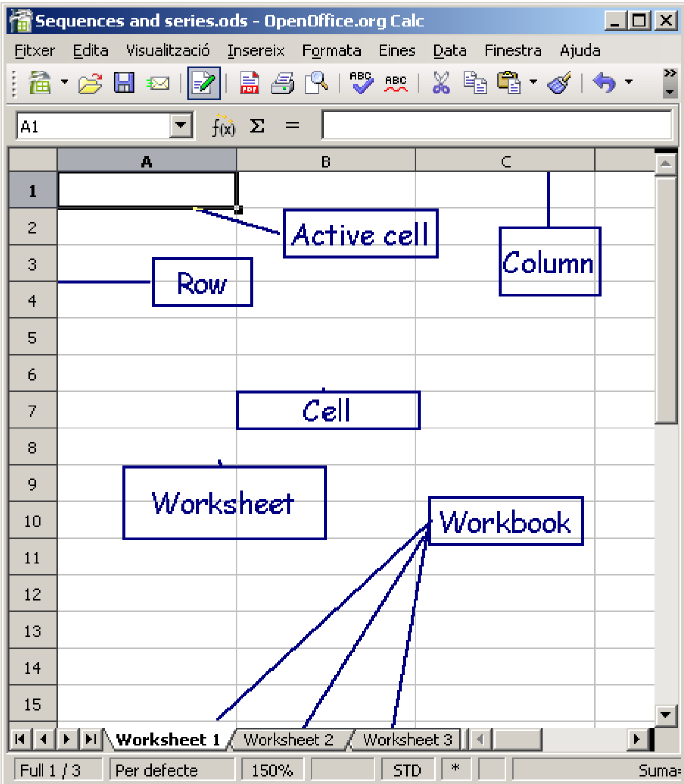
___________ always starts with "=" signs and the calculations for each cell

___________ is the column number and the row letter of a cell

___________ is each individual box on the spreadsheet

___________ is one page of a spreadsheet

___________ is a set of worksheets



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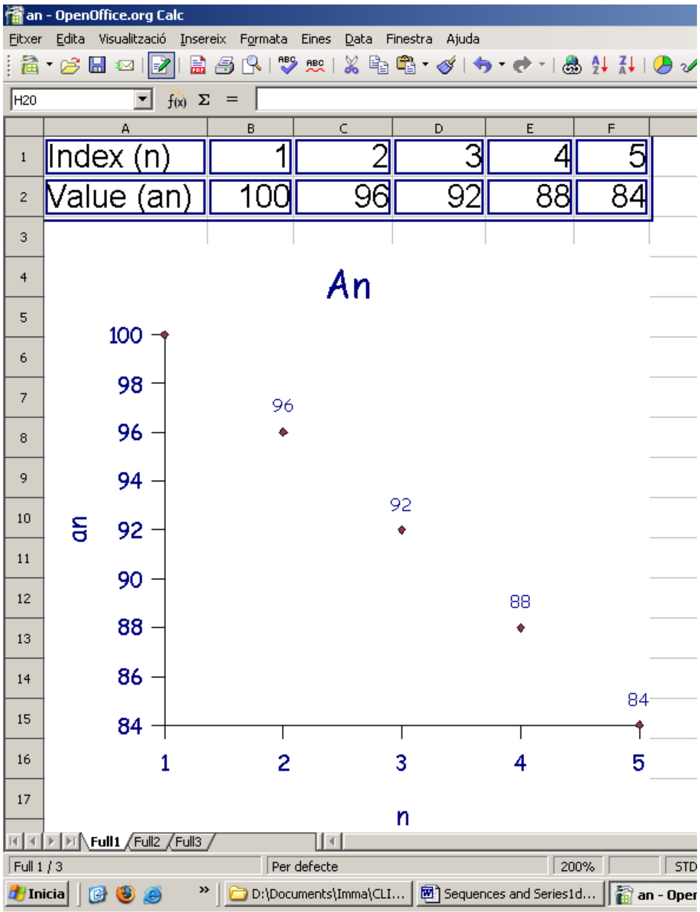
14) Sequences where the rule is to add or subtract the same amount each time are linear sequences. Do you know why?

Use the Calc Spreadsheet to represent the terms of two sequences from exercise 11. I give you one example!!



Sequences where the rule is to add or subtract the same amount each time are named **linear sequences** and their graph is like …

- ☐ a linear function
- ☐ a parabola

Definition 4 ⇨ A sequence is a function f from a subset of the integers (usually N or Z+) to a set S. We often write an = f(n). Each a_n is a term of the sequence.

Fill in the gaps and observe that a sequence is like a function!

a1 = f(1)=100 a3 = f(__)=__ a__ = f(5)=__

a2 = f(2)=96 a__ = f(__)=88

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Fill in the missing numbers or expressions in each linear sequence below

(a) 4, 6, __ , __ , __ , 14,...

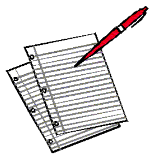
(b) x, x + 4, _____ , x + 12, ...

(c) 1, __ , 11, __ , 21, __ ,...

(d) n – 5, n – 3, n – 1, __

(e) a, a + b, a + 2b, _____ , ...

Experiment 1: Making power with our hands



(1) In pairs take a piece of paper and cut it in half.

(2) Put the pieces together and cut them in half

(3) Repeat this action six times and complete:

Cuts:	1	2	3	4	5	6
Pieces of paper	2	4				

Look at the numbers of pieces, do you think that it is a sequence?__________

Underline the correct word, to describe the sequence:

The sequence of numbers of cuts is written An = 1,2,3,4,5,6.

It is a(n) *finite / infinite* sequence with only *six / seven* terms.

Each term is obtained by *adding / subtracting* one unit to/from the preceding term, so it is a(n) increasing/decreasing sequence.

The terms of that sequence are *odd / natural / even* numbers.

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Now describe the sequence of pieces of paper. You can use the description above.

The sequence is written Bn = 2,4,__,__,__,__

It is a(n) ____

____

Each term ____

____

The terms ____

Can you write Bn as a function? F(n) = ____

- **Thinking challenge!!**

It's impossible to fold a piece of paper more than eight times!

Sounds Odd, doesn't it? What is the reason for that?

Try it yourself and try to answer. Think about the thickness of the paper, the number of layers and the mathematical rule.

I think that the reason for this is that…
I think it is impossible because…
This is due to…

____

____

____

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Alice laughed: "There's no use trying," she said; "one can't believe impossible things." "I dare say you haven't had much practice," said the Queen.

Through the Looking Glass by L. Carroll

Look for Britney Gallivan on the net and write two sentences about her record.

____

____

____

____

____

15) Let's read.

Robin Hood is a folk hero from the Middle Ages. He is a legendary character whom people have told stories about for many years. There are lots of books, plays, movies and cartoons about his story. Do you know who Robin Hood was?

Talk in pairs about Robin Hood and try to write two sentences about him.

____

____

Now read this text in order to find more information:

Robin Hood lived in Sherwood Forest near the town of Nottingham, England. His enemies were Prince John, and the corrupt Sheriff of Nottingham, who abused their powers and took money from the people who needed it. Robin Hood used his ***archery*** *skills and his* ***wit*** *to steal the money back, and return it to the poor. Accompanying Robin were his* ***faithful*** *followers (The Merry Men).*

From http://simple.wikipedia.org/wiki/Robin_Hood

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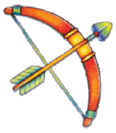

Why did Robin Hood steal from the rich?

Find the answer with the help of sequences!!

Find the missing number in each of the following:

E 2,4,6,_8_,10,12,...

A __,7,9,11,13...

I 3,6,9, __...

B 12,10,8,6,__,...

S 5,10,__,20,...

T 10, 12, 14, __, 18,...

U 1,4,7,10,__,16,...

H 15,12,9,__,3,....

Y 14,10,6,__

C 0,4,8,12,16,__,...

V 10,15,20,__,...

N 25,20,15,__,5,...

D 1,1,2,1,1,2,1,__

O 3,4,3,3,4,4,_,3,3,4,4,4,...

R 10,20,30,40,__,60,...

M 2,6,10,14,__,22,...

P 3,8,13,18,__,28,...

Now complete the secret message and you will find the answer by writing each letter in its right place:

__	E	__	__	__	__	E		__	__	E		__	__	__	__
4	8	20	5	13	15	8		16	6	8		23	3	3	50

__	__	__	__	__		__	__	__	E		__	__	__
1	12	1	10	16		6	5	25	8		5	10	2

__	__	__	E	__
18	3	10	8	2

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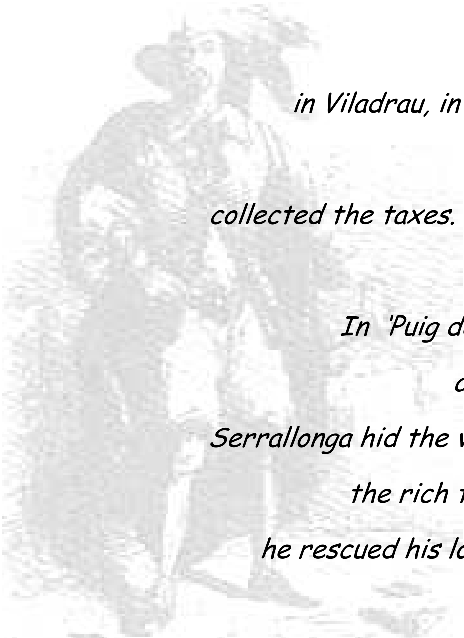
- Robin Hood is an English legendary character. Looking at the next activity you will realize that people in different countries have lots of things in common.

  Do you know who is Serrallonga? Read the text below and try to find the hidden sequence in even and odd lines:



Serrallonga was born → 3 words
in Viladrau, in 1598. He is a mythical Catalan brigand. He used to
assault the royal vehicles that
collected the taxes. In 1627, persecuted by the soldiers of Felipe IV, Catalan
people helped him to flee to France.
In 'Puig de la Força' in the remains of an old castle exist
a buried and dark cell, where it is said
Serrallonga hid the victims of his kidnappings. Legend says that he stole from
the rich to distribute the loot among the poor and that
he rescued his loved one from his enemies who were against the poor.

The sequences I have found are:

Odd lines ___________________________________

Even lines ___________________________________

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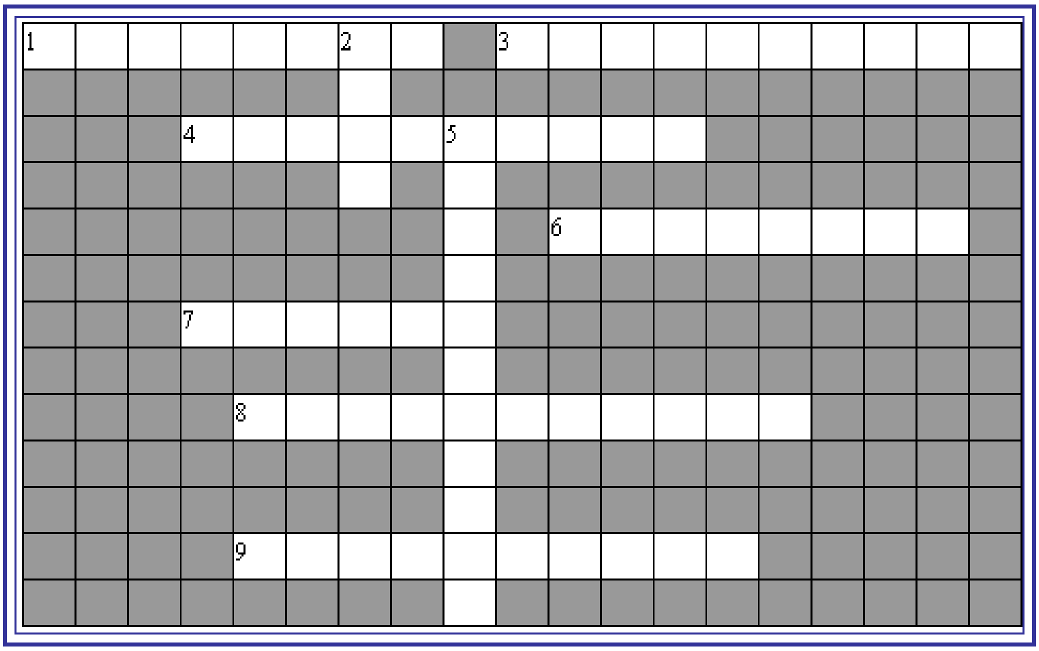
16) Solve this crossword.



ACROSS

1. If a sequence goes on forever, it is...
3. The terms of this sequence drop and drop...
4. The terms of this sequence grow and grow...
6. Ordered list of elements
7. Sequences where the rule is to add or subtract the same amount each time.
8. Catalan legendary character who lived in 1627
9. English legendary character who lived in Sherwood Forest

DOWN

2. Each element of a sequence is named...
5. The terms of this sequence grow and drop..

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Lesson 2: Recurrence

Fibonacci's Rabbits and recurrent sequences

In this activity, you will solve a problem posed over 800 years ago. An Italian mathematician, Fibonacci, in 13th century bred rabbits, and he wanted to sell them until he had enough money to buy a boat. So first he bought a pair of rabbits.

He had to calculate how long it would take to save enough money for his boat and from this came a famous problem:

- A pair of rabbits cannot bear young until they are two months old. But once a pair reaches maturity, they will produce one new pair of rabbits each month. If you start with one pair of newborn rabbits, how many pairs of rabbits will you have at the beginning of each month there after?

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1) Use the space below to record several strategies for solving this problem.

Discuss your strategies with a partner and fill the gaps. Start to count in January!

To get the next number we …	multiply…
	divide …
	add…
	subtract…

____

____

____

____

We think that in January there *was/were* ____ pair(s) of rabbits, in February there *was/were* ____ pair(s) of rabbits, in March there *was/were* _____ pair(s), in April _____ pairs and so on.

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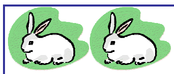
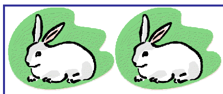
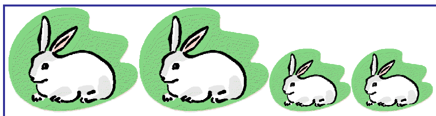
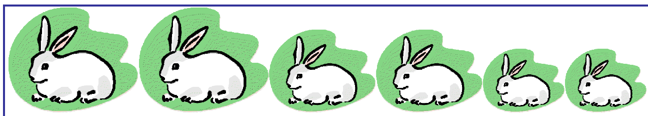
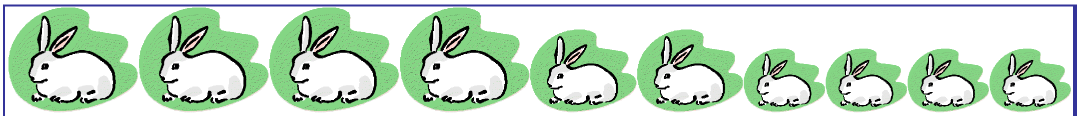
The chart below can help you. Can you complete it? Don't worry if you are not a good artist, you can just draw the ears of the Rabbit!

Month	Pairs of Rabbits	Pairs
January		1
February		1
March		2
April		3
May		5
June		
July		
August		

When Fibonacci solved this problem he discovered a special number pattern: 1, 1, 2, 3, 5, 8, 13, 21....

The **Fibonacci sequence** has fascinated people ever since. These numbers keep appearing in unexpected places, like the numbers of petals on flowers, and in your hands… You have 2 hands, each of which has 5 fingers, each of which has 2 or 3 parts, separated by 1 or 2 knuckles! Have a look at http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibnat.html for lots more!

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As you know we use an algebraic notation for sequences, An= a_1, a_2, …
With this same notation, we would write $\mathbf{a_n}$ to represent the **nth term** in the sequence, An= a_1, a_2, …, a_n

Definition 1 ⇨ The **Fibonacci sequence** is an infinite sequence where each term (from the third term onwards) is obtained by adding together the two previous terms.

To represent the rule for the Fibonacci sequence, we would write the nth term as

$$\boxed{an = a_{n-1} + a_{n-2}}$$

to say that each term was the sum of the two preceding terms. Notice that $a_{n-1} \neq a_n - 1$

Definition 2 ⇨ A **recurrent sequence** is a sequence where each term is defined as a function of the preceding terms. The Fibonacci sequence is recurrent.

2) Look at this sequence: An=2, 5, 11, 23, 47, … it is a recurrent sequence. What is its recurrence relation? Try to make a sentence that answers the question.

| previous | adding 1 | the | double of | the | term. | to |

Each term (from the second term onwards) is obtained by …

____

What is the nth term? an = _____+ 1.

To be more specific, if you have to give one term in the sequence, you almost always give the first term, a1.

an = ______ ; a1=___

Now check each term:

a_1=___ a_4=___

a_2=___ a_5=___

a_3=___ a_6=___

3) Find the recurrence relation of the sequences below:

An= 5,8,11,14,17,... Bn= 2,9,16,23,30,...

4) A sequence is given by the formula

- an = 3n + 5, for n = 1, 2, 3, ... Write down the first five terms of this sequence.

- an = $1/n^2$, for n = 1, 2, 3, ... Write down the first four terms of this sequence. What is the 10th term?

5) Write down the first five terms of the sequence given by

an = (–1)· (n+1)/n.

a_1=__________=_____ a_3=__________=_____ a_5=__________=_____

a_2=__________=_____ a_4=__________=_____

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Not all sequences have an nth term with a mathematic expression.

6) Work in pairs. Match each sequence with its nth term. Be careful, there are two sequences that don't have an nth term!

1,2,3,4,5,6,7,8,9,10,...	$a_n= 2^n$
2,4,6,8,10,12,14,16, ...	decimal places of pi
1,3,5,7,9,11,13,15,...	$a_n= n$
2,4,8,16,32,64,128...	$a_n=2n-1$
1,4,9,16,25,36,49,...	$a_n=2n$
1,2,3,5,7,11,13,17,...	prime numbers
1,4,1,5,9,2,6,...	$a_n= n^2$

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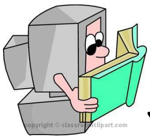
7) Can you give the next term of each of these sequences? Can you give the nth term that defines the sequence? Are they recurrent? Assume the first term is a_1.

- 1/2, 1/3, 1/4, 1/5, 1/6, …
- 7, 8, 9, 10, 11, 12, …
- 1, 3, 6, 10, 15, 21, …
- 1, 1, 1, 1, 1, 1, 1, …

The most important types of **recurrent sequences** are the **Fibonacci** sequence and the **arithmetic and geometric progressions**. We are going to look at progressions in the next lesson.



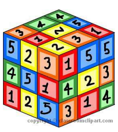
You are now going to use the computer to generate and represent sequences graphically. They must include ascending, descending, constant and periodic sequences. Also try and use sequences that include negative numbers.

- Generate three different sequences and 10 terms in each one. Then omit 2 numbers and give the sequence to your partner to solve.
- You can easily calculate many of the terms of Fibonacci's sequence in Calc. You will find about 40 terms for the sequence!

The following explanation shows you the process you have to follow to find the Fibonacci terms, but… the steps in the process are out of order!

8) Work in pairs. Calculate the third term of the sequences whose nth term is given by the expressions below. After that sort out the sentences looking at the number of the sentence and you will obtain the ordered text!

1. $a_n=4\cdot(n^2-2)$ $a_3=$____
2. $a_n=7n-5$ $a_3=$____
3. $a_n=n(n+1)$ $a_3=$____
4. $a_n=3+n^2-n$ $a_3=$____
5. $a_n=6n-n^3$ $a_3=$____
6. $a_n=7-3n$ $a_3=$____



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(9) In B4 enter the formula =B2+B3 Now press 'enter', and your third term will be calculated.

(16) Select these first three numbers and drag them downwards.

(12) Go to column B and enter first a few terms of Fibonacci's sequence. (1 in B2 and 1 again in B3).

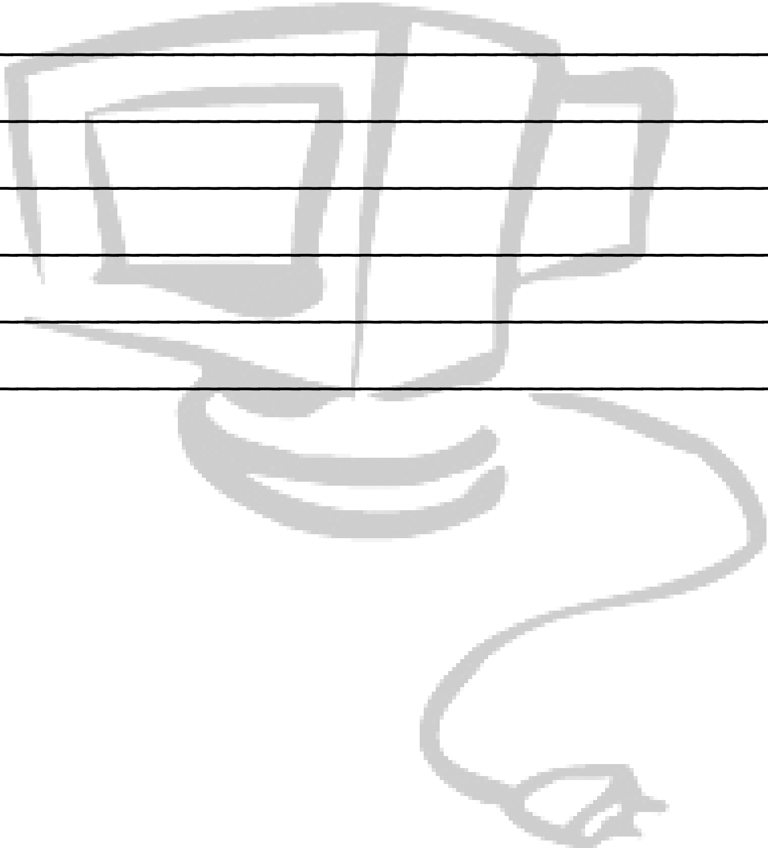
(28) In column A enter numbers from 1 to 3.

(-9) By dragging the formula from cell B4 downwards you will get other terms in Fibonacci's sequence.

(-2) The computer will recognise the pattern and continue with other counting numbers.

- To find the Fibonacci terms you have to follow these instructions:

1. ____
2. ____
3. ____
4. ____
5. ____
6. ____



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9) Let's read.

Below you have the slides Professor Langdon projected to his students. Langdon is one of the main characters in Dan Brown's book "The Da Vinci Code"

In pairs try to match the words with the appropriate image.

1. Nautilus

2. Vitruvian Man

3. Stradivarius

4. Honeybee

5. Sunflower

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Work in groups of 4. Give out each part of the text and read it carefully. Underline the words above with a colour in your text and the sentence in which they appear. When you finish, with your group try to answer the questions in the next exercise,

Student 1

" Number PHI, 1,618 , is a very important number." Langdon said, "And it is considered the most beautiful number in the universe." He explained that PHI was derived from the Fibonacci sequence because the quotients of adjacent terms approach the number PHI. "PHI is very important in nature" said Langdon " so the ancients assumed that it came from the Creator of the universe and scientists called PHI the Divine Proportion."

Student 2

"In a honeybee community, if you divide the number of female bees by the number of male bees you always get the same number, PHI."

Langdon smiled and he projected a slide of a spiral seashell, a nautilus. "The ratio of each spiral's diameter to the next is PHI. Another example, sunflower seeds grow in opposing spirals, and the ratio of each rotation's diameter to the next is PHI"

Student 3

Langdon pulled up another slide The Vitruvian Man. " Da Vinci was the first to show that the human body is full of proportional ratios related to PHI. Don't believe me? Measure the distance from the tip of your head to the floor. Then divide that by the distance from your belly button to the floor. Guess what number you get. Yes, PHI, Want another example? Measure the distance from your shoulder to your fingertips, and then divide it by the distance from your elbow to your fingertips. PHI again. Another? Hip to floor divided by knee to floor. PHI again. My friends, each of you are a walking tribute to the Divine Proportion."

Student 4

Langdon explained that PHI appeared in the Greek Parthenon, the pyramids of Egypt… in Mozart's sonatas, Beethoven's Fifth Symphony, as well as the works of Bartók, Debussy, and Schubert and was even used by Stradivarius in the construction of his famous violins.

Answer the following questions:

- Why is the number Phi important?

____

____

____

- Can you demonstrate that Phi derives from the Fibonacci sequence? Calculate this number according to the text.

____

____

- Write down two or three examples of Phi in nature, in architecture and in music.

____

____

____

____

- With a tape measure, try to prove that the human body is a tribute to the Divine Proportion.

- This number turns up in all kinds of places and in all kinds of ways. Its value is

$$\phi = \frac{1}{2}\left(\sqrt{5}+1\right)$$

- Use your calculator to calculate its approximate value. Remember, $\sqrt{5}$ is an irrational number, which means that you can not find its precise value on the calculator (or in any other way!).

- Open the spreadsheet you made in exercise 9 and you will get the approximations for φ in column C by entering another formula.
- Observe the values obtained for φ. Analyse these values and in pairs try to explain what happens.

We notice that…

the further the terms go the values for φ	are are not	more and more accurate. bigger. smaller.

φ is	an irrational number a rational number	and	doesn't have has	an exact value an approximate value

whatever terms of the Fibonacci's sequence we chose, we	always never	obtain approximations

- Write three sentences

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10) Look at these two pictures. Describe them. Have they something in common?

'Fibonacci's World'. Mary Anne Durnin

'Semitaza gigante volante, con anexo inexplicable de cinco metros de longitud'. Salvador Dalí,

<table>
<tr><td>In the middle</td><td rowspan="5">of the picture</td><td rowspan="2">there is/ are.....</td></tr>
<tr><td>In the top/bottom left hand side/corner</td></tr>
<tr><td>In the top/bottom right hand side/corner</td><td>colours are....</td></tr>
<tr><td>In the foreground</td><td rowspan="2">I can see.....</td></tr>
<tr><td>In the background</td></tr>
</table>

- Look at the titles. What do they suggest?

  I think......

  In my opinion...

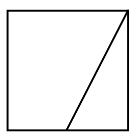
  I suppose the artist wants to…

A golden rectangle is a rectangle the lengths of whose sides are in the golden ratio, that is, approximately 1:1.618. Work in pairs and try to find the golden rectangle in each of the paintings. How many golden rectangles can you see?

Now it is your turn. Read how to construct a golden rectangle and after that… be an artist and in pairs try to make a Fibonacci picture. They will be used to decorate our classroom.

You will need a ruler and a compass

- Construct a square
- Draw a line from the midpoint of one side of the square to an opposite corner



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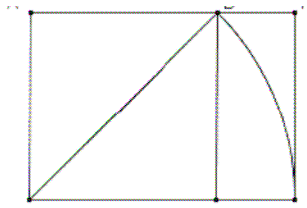
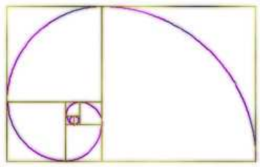
- Use that line as the radius to draw an arch that defines the height of the rectangle
- Complete the golden rectangle



You can use the Golden Rectangle to draw this beautiful spiral



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Arithmetic and Geometric Progressions

Lesson 3 : Basics of Arithmetic and Geometric sequences

Arithmetic Progressions (AP)

Definition 1 ⇨ An **arithmetic sequence or progression (AP)** is a sequence whose consecutive terms have a **common difference**.

1) See if you can identify the arithmetic progressions from the progressions below and find the common difference in the A.P.s:

A. 5, 7, 9, 11, 13, 15, ...

B. 1, 11, 21, 31, 41, 51, ...

C. 2, 4, 8, 16, 32, 64, ...

D. 1, 1.5, 2, 2.5, 3, 3.5, ...

E. 9, 6, 3, 0, -3, -6, -9, -12, ...

2) What is the value of 10^{th} term in the APs below?

A. 5, 8, 11, 14, 17, ... The value of the 10^{th} term is…

B. -6, 0, 6, 12, 18, 24, ... ______________________________

C. 1, 3, 5, 7, 9, ... ______________________________

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The nth term of an AP

The n^{th} term of an AP is given by the rule

$$\boxed{a_n = dn + c \qquad \text{or} \qquad a_n = a_1 + (n-1)d}$$

where d is the common difference and c is a constant.

Observe: if we write a_n as $a(n) = dn + c$ it looks like the familiar linear function whose rule is usually presented as $f(x)=mx+n$.

In Lesson 1 we introduced a new definition of sequence (Definition 4), read it and then try to make an equivalent definition of an arithmetic sequence:

An arithmetic sequence is a function whose rule may be expressed as …

f(n) = dn + c	is the common difference	c	d
of the form	a linear equation	f(1) – c.	and
where	is a constant		

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3) Work in groups. Compare the equivalent definition of an arithmetic sequence with the definition of a linear function to conclude the following.

The domain of	a linear function is	R
	an AP is	N

R⇨ Real numbers. N⇨ Natural numbers.

The rule for	a linear function is	$f(x)=mx+n$
	an AP is	$a_n=dn+c$

The number b in	a linear function is	the range value associated with 0
	an AP is	the y-intercept

4) The AP a_n has a common difference 6 and the first term is -5. Find the rule.

5) If an AP has the common difference 3 the fourth term is 14, what is its rule?

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6) If the fifth term of an AP is 14 and the eighth 20, what is its rule?

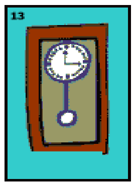
7) The rule for an AP has the form $a_n = 6n + c$. The fourth term is 10, what is the value of c?

8) What is the nth term of each of these sequences?

4, 7, 10, 13, 16, 19, … -4, -2, 0, 2, 4, 6, …

'Alice heard the Rabbit say to itself, `Oh dear! Oh dear! I shall be late!' (all seemed quite natural); but when the Rabbit actually took a watch out of its waistcoat-pocket, and looked at it, and then hurried on, Alice started to her feet, for it flashed across her mind that she had never before seen a rabbit with either a waistcoat-pocket, or a watch to take out of it…'

9) A clock strikes the number of times of the hour. How many strikes does it make in one day?



For the first 12 hours of the day, the clock will strike … times

For the next 12 hours, there will be … strokes.

In one day, the clock will strike … times.

Are the hours an AP? Do you remember the name of sequences that revisit the same values over and over again?

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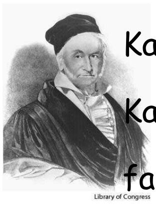
Gauss a Child Mathematical Genius



Karl Friedrich Gauss was born in Brunswick, Germany in 1777. It is said that Karl displayed incredible talent in maths at a very young age. He helped his father in business accounts before the age of 5 and one day he surprised his maths teacher. The teacher asked him to add up the numbers between 1 and 100, (to keep him busy, he was a naughty student). Gauss quickly found a short cut to the answer 5050.

Do you know how he managed it?

TRICK

- Write down the series forwards and backwards.

  $S_{100} = 1 + 2 + 3 + \ldots + 98 + 99 + 100$

  $S_{100} = 100 + 99 + 98 + \ldots + 3 + 2 + 1$

- Now just add S_{100} to S_{100}. Fortunately for Gauss, and us,

  $1 + 100 = 2 + 99 = 3 + 98 = \ldots = 97 + 4 = 98 + 3 = 99 + 2 = 100 + 1 = 101.$

  Then,

  $S_{100} = 1 + 2 + 3 + \ldots + 98 + 99 + 100$

  $S_{100} = 100 + 99 + 98 + \ldots + 3 + 2 + 1$

  $2\ S_{100} = 101 + 101 + 101 + \ldots + 101 + 101 + 101$

- All these pairs of numbers add up to 101!

  Then, $2\ S_{100} = 101 + 101 + \ldots + 101 + 101$

  They have to be ____101s here so

  $2\ S_{100} =$ ___ × 101 = _____ ⇨ $S_{100} =$______

10) Use the Gauss trick to find the Sum of the Natural Numbers

$$1, 2, 3, 4, 5, \ldots, n$$

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Definition 2 ⇨ An **arithmetic series** is an A.P. where we ADD each term of the A.P. In other words, if you look at the A.P. of hours and replace the commas with plus signs you get:

$$1+2+3+4+5+6+7+8+\ldots$$

Imagine trying to add ALL the terms in the sequence of odds numbers! You couldn't do it. (It would add up to infinity.) However, you WILL be asked to add up a set number of terms in a series. The formula to help you do this is:

$$\boxed{S_n = \frac{n}{2}(a_1 + a_n)}$$

11) Add the first 10 terms of the A.P. shown here: 3, 11, 19...

n =____ a_1=____ a_{10}=____

S_{10}=

In mathematics, the Greek letter sigma Σ means the sum of.

$$\sum_{k=1}^{10} = a_1 + a_2 + \ldots + a_{10}$$

The above symbol is telling you to find the sum of each term from term 1, to term 10 (again, replacing k with 10 to get??).

12) Find the sum of the first five terms of the AP with first term 3 and common difference 5.

Geometric Progressions (GP)

Definition 3 ⇨ A **geometric sequence or progression (GP)** is a sequence where each new term after the first is made by multiplying the previous term by a constant amount.

13) See if you can identify the geometric progressions from the progressions below and find the constant amount or common ratio in the GP.s:

A. 1, 3, 9, 27, . . .
B. 1, 11, 21, 31, 41, 51, ...
C. 2, 6, 18, 54, ...
D. 1, 1.5, 2, 2.5, 3, 3.5, ...
E. 1, −2, 4, −8, . . . ,

14) What is the value of the common ratio in the GPs below? And the 5th term?

A. 1, 1/2 , 1/4, . . .

The value of the common ratio is... and the 4th term ...

B. −6, 18, . . .,

____

C. 3, 12, 48, ...

____

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The nth term of an GP

The n^{th} term of a GP is given by the rule

$$\boxed{a_n = a_1 r^{n-1}}$$

where r is the common ratio.

Observe: if we used normal functional notation the rule would be written as **a(n) = a_1r^{n-1}** which looks like an **exponential function** whose rule is normally presented as **f(x) = ar^x**.

Definition 2 ⇨ A **geometric series** is a GP where we ADD each term of the G.P. In other words, if you look at the G.P. of pieces of paper in Lesson 1 and replace the commas with plus signs you get:

$$2+4+8+16+32+\ldots$$

Imagine trying to add 30 of the terms in the sequence of powers of 2! It is so hard! The formula that can help you find the **finite sum** is:

$$\boxed{S_n = \frac{a(1-r^n)}{1-r}}$$

where r is the common ratio a the starting value (a_1) and provided that $r \neq 1$.

15)Add the first 6 terms of the G.P. shown here: 2, 6, 18, ...

n =____ a=____ r=____

S_6=

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16) Find the sum of the first five terms of the GP with first term 3 and common ratio 5.

17) Find the sum of the first five terms of the GP 8, – 4 , 2 ,– 1 . . .

18) Work in pairs. How many terms are there in the geometric progression

2, 4, 8, . . . , 128 ?

Imagine trying to add ALL the terms in the sequence of a GP! You couldn't do it. (It would add up to infinity.) **The sum to infinity** or the **infinite sum** of a geometric progression with starting value a and common ratio r is given by

$$\boxed{S_\infty = \frac{a}{1-r}}$$

where $-1 < r < 1$. Notice that if n is infinite and $|r|<1$, then $r^n=0$.

19) Find the sum to infinity of the GP with first term 3 and common ratio 1/2 .

20) The sum to infinity of a GP is four times the first term. Find the common ratio.

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` For some minutes Alice stood without speaking, looking out in all directions over the country -- and a most curious country it was. There were a number of tiny little brooks running straight across it from side to side, and the ground between was divided up into squares (…). She declares: it's just like a large chessboard! It's a great huge game of chess that's being played -- all over the world (…) How I WISH I was one of them! (…) of course I should LIKE to be a Queen, best.'

"Through the Looking Glass"

The wheat and the chessboard

One of the earliest mentions of Chess in puzzles is by the Arabic mathematician Ibn Kallikan who, in 1256, poses the problem of the grains of wheat, 1 on the first square of the chessboard, 2 on the second, 4 on the third, 8 on the fourth etc.

How many grains of wheat are there?

Try to organize your work.

In pairs decide your strategy and write it down. Use the charts below.

In a chessboard there are…squares.

We need to find…

First we … second…

Finally…

The sequence of grains of wheat is…

We have to use the formula…

The first term in the sequence is…

The common ratio is…

____

____

____

____

____

____

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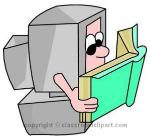
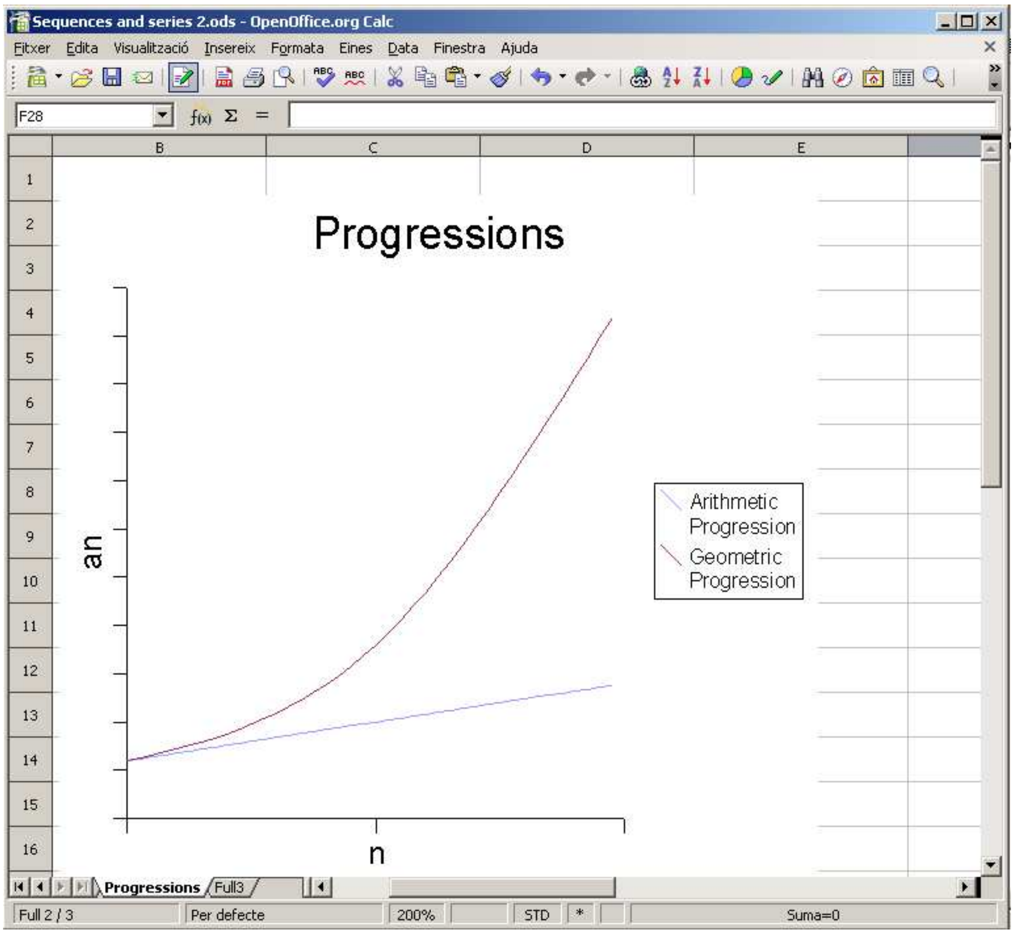


Graph of Arithmetic and Geometric Progressions You are now going to use the computer to generate and represent progressions. Choose an arithmetic and a geometric progression and represent them in a Calc Spreadsheet. I give you an example.



What are the differences between the two graphs? Discuss with a partner.

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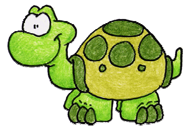
The arithmetic/geometric progression	rises goes up increases	slowly gradually fast sharply suddenly considerably a lot very much dramatically
	falls goes down decreases	

The slope	stays remains	the same constant steady stable
	fluctuate	
	is/are	fluctuating irregular inconstant unsteady unstable

Make comparisons with a partner.

….	is are	greater than	…
….		smaller than	

Zeno's Paradox of the Tortoise and Achilles



Zeno of Elea (circa 450 b.c.) is credited with creating several famous paradoxes. One of these is the paradox of the Tortoise and Achilles. (Achilles was the great Greek hero of Homer's The Illiad.) This paradox has inspired many writers and thinkers, such as Lewis Carroll and Douglas Hofstadter, who also wrote dialogs between Achilles and the Tortoise.

In groups of three look for this paradox on the Internet and try to solve it with the help of a series. Can the tortoise win?

One problem in economics is the calculation of the **present value** and the **compound interest**. Imagine you invest an amount of money C_i for n years at a rate of interest of i% (where interest of "5 percent" is expressed fully as 0.05) compounded annually, the present value of the receipt of C_f, n years in the future, is:

$$\boxed{C_f = C_i(1+i)^n}$$

21) In pairs compare this formula with the nth term of a GP, and then solve this problem.

- When Robert was born, his father invested €500 at a rate of compound interest of 9%. How much money did Robert have in his 20th birthday? Give the answer in €, £ and in $.
- Look for the currency exchange rates on the Internet, and write the name of different countries where you have to use Euros, Dollars and where you use Pounds. What language do they speak?

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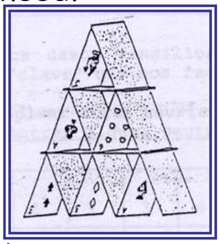
Experiment 2: breaking The Guinness world Record

To build this card castle, how many cards do we need?



____

How many cards do we need to build a 10-floor castle?

Use the space below to record several strategies for solving this problem.

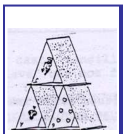
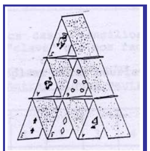
Discuss your strategies with a partner.

____

____

____

The chart below can help you. Can you complete it?

Card castle					
Number of floors	1	2	3	4	5
Number of cards	2	7	15		

The Guinness world Record is a 61-floor castle. How many cards do you need to break the record? Try to give an answer by using a table or a graph.

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