LESSON PLAN 3.The Sierpiński triangle

Aim: To introduce pupils to one of the simplest examples of the geometric objects known as fractals, created by following a specific set of rules. Rules involve dividing an image into smaller pieces similar to the original and then removing some of those pieces. One of the formal properties of fractals, self-similarity, appears.

Teaching objectives:

Content
- Geometric iteration rule by removals
- Copies of copies method
- Areas and perimeters
- Numerical and geometric patterns
- Geometric transformations. Central symmetry
- Self-similarity

Communication
- Predicting: “tends to ...”, “will be...”
- Comparing: “getting smaller ...”, “getting larger ...”, “magnified by a factor of ...“, “reduced by ...”
- Describing your work: “I have chosen ...”, “I have reduced ...”, ...
- Arguing with partners predictions and conclusions
- Translating visual information into verbal descriptions
- Checking calculations in pairs
- Sharing opinions to organise a poster
- Reading summary sentences to other groups
- Writing a script for an oral presentation

Cognition
- Identifying similar shapes
- Predicting eventual outcomes
- Calculating parts of a unit whole, perimeters and areas
- Creating figures through drawing programs
- Creating spreadsheets to display data
- Plotting a graph to illustrate selected information
- Writing summary reports using key vocabulary
- Arguing and checking answers
- Using Internet resources to confirm predictions

Context
- Zooms of the size of images
- Computer graphics and visual art
- The chaos game
- History of Mathematics

Outcomes: At the end of the lesson, pupils will be able to

- Distinguish between two similar figures and a self-similar figure
- Identify numerical and geometric patterns
- Translate visual information to verbal descriptions
- Use drawing programs to generate further iterative outputs
- Predict what will happen if we continue iterating
- Confirm predictions using the Internet resources
- Be confident in geometric English language
- Develop a sense of responsibility when working in a cooperative group
- Recognise the importance of arguing in a team
- Know how to behave in a discussion
- Know how to make a script for an oral presentation
- Present their work to the rest of the class

Key words: geometric iteration rule, infinity, Sierpiński triangle, self-similarity, Pascal triangle, Chaos game.

Prerequisites: Pupils need to be familiar with equilateral triangles, regular tetrahedra, whole powers and fractions as a part of a whole.

Curriculum connections:

- Calculation of perimeters and areas.
- Computation with fractions in a geometric context.
- Triangle's geometry.
- Practise with recognizing and describing numerical and geometric patterns.
- Geometric transformations. Similarity.
- Cartesian coordinates and graph plotting.
- Use of calculator or computer when needed.
- Pascal triangle.
- Tetrahedra.

Timing: Nine 55-minute class periods.

Tasks planned:

First session

- Pupils work through exercise 1-*Removals* using pencil, ruler and coloured pencils. A geometric iteration rule must be familiar from the previous unit, so have them practise on their own. Only circulate among them to be sure they are headed in the right direction.
- Don't correct their predictions. Pupils can do it themselves after having sketched the fourth iteration.
- Plenary: sharing and discussing solutions.
- Homework: Ask to finish accurately the exercise.

Second session

- Provide pupils with worksheet 2- *Sierpiński*. They go to the indicated Web site and look carefully at the eighth iteration of the first session exercise.

- Pupils work through this exercise. Emphasize those iterations are approaching the Sierpiński triangle: none of them is the Sierpiński triangle itself.
- Ask pupils to start working with exercise 3-*Area of the Sierpiński triangle* and to finish it as homework.

Third session
- Plenary: correction of exercise 3 and discussion about it.
- Pupils work through exercise 4-*Perimeter of the Sierpiński triangle.*
- Plenary: correction of exercise 4 and discussion about it.

Fourth session
- Pupils work through exercise 5-*Copies of copies* by using a computer drawing program, familiar from the previous lesson, to sketch again the first four iterations approaching the Sierpiński triangle.
- All of the class cooperates by making a poster following the instructions from exercise 6-*Poster presentation*.
- Homework: Ask pupils to answer the questions from exercise 5.

Fifth session
- Divide the class in half and have one group of pupils working through exercise 7-*Triplicating a circle* and the other group through exercise 8-*Triplicating a square.*
- Each group prepares a poster and writes a script for an oral presentation.
- Plenary: poster presentation and oral process explanation.
- Homework: Ask pupils go one step further by *Triplicating any image* (for example, their own picture) and making a poster with it.

Sixth session
- Each pupil presents to the rest of the class his/her home project.
- Provide pupils with worksheet 10-*Self-similarity* and remain them of the definition of similar figures and what is meant by a ratio of similarity.
- Ask pupils to identify self-similar figures and magnification factors through exercise 10.
- Homework: exercise 11-*Pascal triangle* shows the relationship between the Pascal triangle and the Sierpiński triangle.

Seventh session
- Homework revision.
- Give time to understand the random game in exercise 12-*The Chaos game* and to provide a half-ruler to play hands on.
- Pupils play the game in pairs.
- Homework: running simulations of the game on the Web site.

Eighth session
- All the amount of information from all the pupils is showed altogether.
- Go through *Playing the game with the Sierpiński triangle showing.*
- Plenary: correction of the exercise and comments on it.
- Final conclusion: teacher helps pupils to summarise how a random process can generate a non- random pattern.

Ninth session

- Provide pupils with worksheet 13-*The Sierpiński Tetrahedron*, ice-cream sticks and glue.
- Focus this session on the construction of a Sierpiński Tetrahedron as big as possible. All of the class takes part in it.
- The questions of the exercise must being answered meanwhile.

Resources

- For class activities and homework: the pupils' worksheets and access to a computer drawing program.
- For drawings and making posters: ruler, pencil and rubber, coloured pencils, cardboard, scissors and glue, half-ruler, pen, markers ...
- For building a *Sierpiński Tetrahedron*: ice-cream sticks and glue.
- For teaching instructions and answer keys to each exercise: the teachers' worksheets, mainly borrowed from the book *Fractals* by J. Choate, R.L. Devaney, A. Foster, listed in the bibliography.

Assessment

- The teacher will take notes of pupils' progress every period.
- Good presentation in worksheets and active participation in the classroom is essential.
- Posters will be individually marked depending on participation, contribution to the group work and oral presentation.
- Homework will be suitably marked.
- Pupils who talk in English will be given extra marks.

Evaluation

TEACHER'S NOTES AND ANSWERS

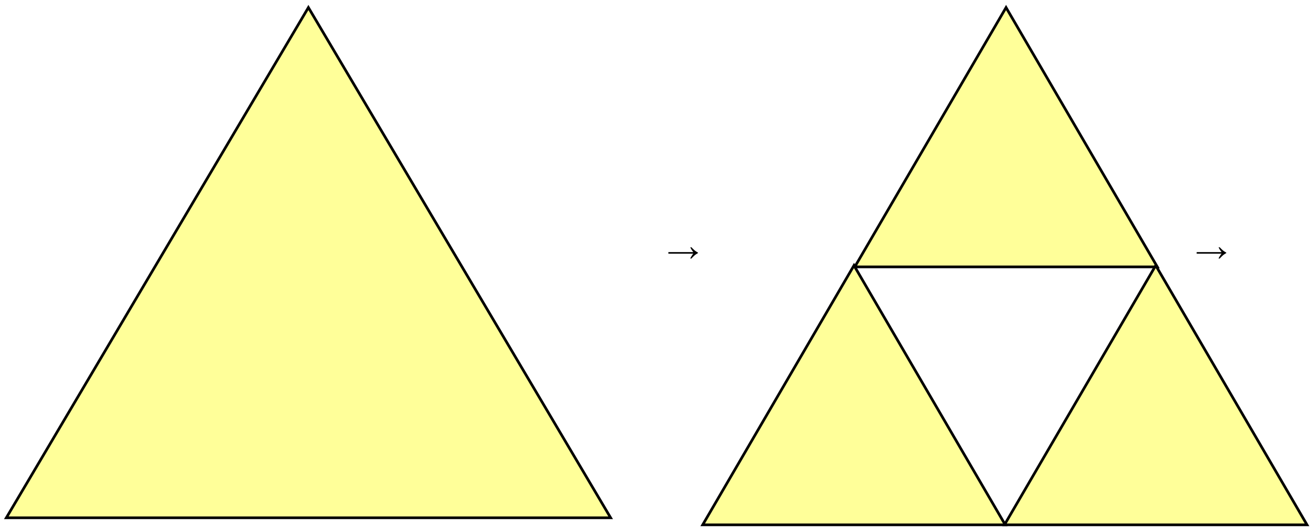
Answers to exercises

1-Removals

Start with a filled equilateral triangle
Apply the iteration rule:

- Divide it into four equilateral triangles by marking the midpoints of all three sides and drawing lines to connect the midpoints.
- **Remove** the **interior** of the triangle from the middle.

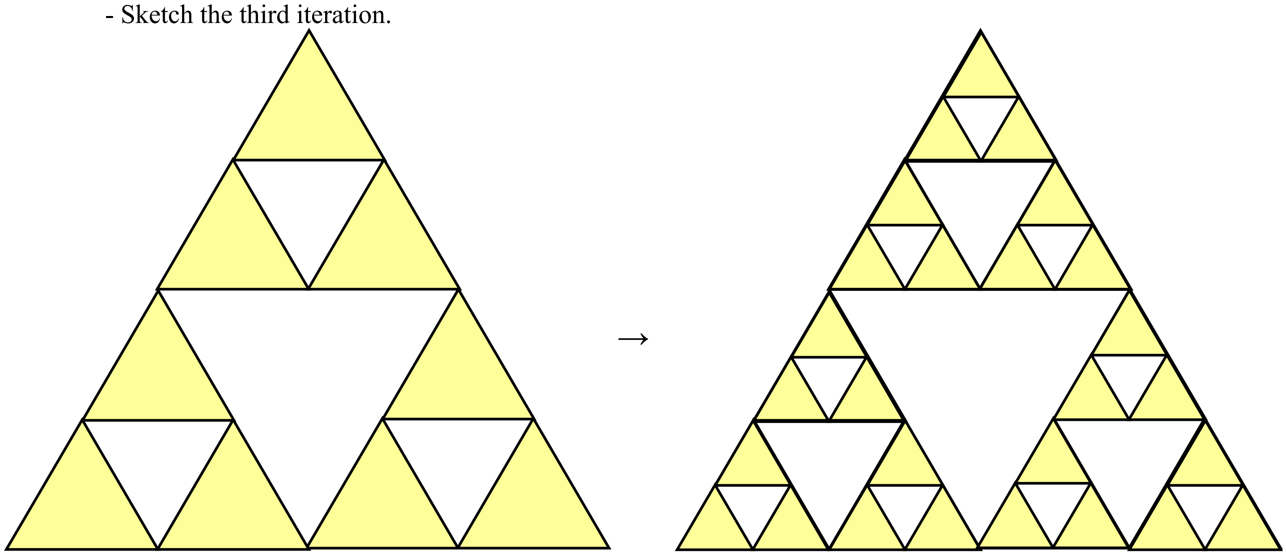
1. Sketch the first iteration. You will have one **hole** and three smaller **filled** equilateral triangle **remaining**.



First iteration

2. - Sketch the second iteration. You will have to apply three times the iteration rule, once on each of the smaller triangles.

   - Sketch the third iteration.



Second iteration Third iteration

3. - **Look carefully at** your sketches and complete the grid below:

	Number of holes **added**	Total number of holes	Total number of triangles **remaining**
Original square	**0**	**0**	**1**
First iteration	**1**	**1**	**3**
Second iteration	**3**	**1 + 3 = 4**	**3 · 3 = 9**
Third iteration	**9**	**4 + 9 = 13**	**9 · 3 = 27**

- Check your answers with your partner.

4. Predict the next row of the grid:

If some pupils are wrong about predictions, don't correct them at this step. After drawing the fourth iteration at the end of the exercise, they can go back to this point.

Fourth iteration	**27**	**13 + 27 = 40**	**27 · 3 = 81**

5. What fraction of the triangle is the total number of triangles remaining in the first iteration?

3 / 4

6. What fraction of the triangle is the total number of triangles remaining in the second iteration? **Pay attention** that holes have different sizes.

9 / 16

7. What fraction of the triangle is the total number of triangles remaining in the third iteration?

27 / 64

8. Do you see a **pattern** here? Use this pattern to predict what fraction of the triangle **would be** the total number of triangles remaining in the fourth iteration.

$(3 / 4)^2 = 9 / 16$ $(3 / 4)^3 = 27 / 64$ $\underline{(3 / 4)^4 = 81 / 256}$

9. Write the fractions in the above questions in order form **greatest** to **least**.

$3 / 4 < (3 / 4)^2 < (3 / 4)^3 < (3 / 4)^4$

10. Assume the **side length** of the original triangle is **1 unit**.

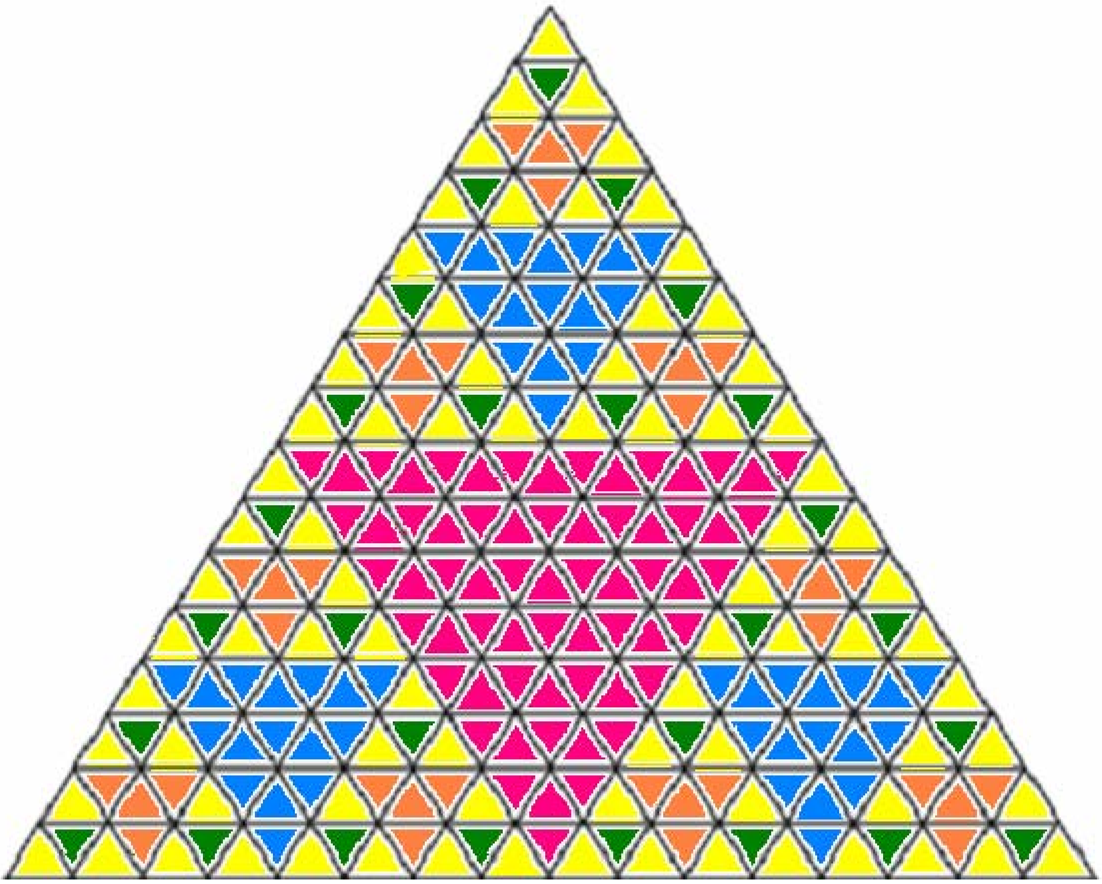
- Look carefully at your sketches and complete the grid below:

	Original triangle	First iteration	Second iteration	Third iteration
Side length of each triangle remaining (as a fraction)	1	**1 / 2**	**1 / 4**	**1 / 8**

- What **fraction** would be the side length of each triangle remaining on the fourth iteration?

1 / 16

11. Use coloured pencils to sketch below how the fourth iteration looks like.



12. **Confirm** predictions you made before about
    - total number of holes and total number of triangles remaining.(4) **40 i 81**
    - side length of each triangle remaining. (10) **1 / 16**

2-Sierpiński

1. Go to: http://math.rice.edu/~lanius/fractals/sierjava.html

a) Play the game and move from one iteration to the next.
The number of holes at each iteration is shown in the corner.
Stop at the 8th iteration (the number of triangles increases rapidly and a big amount of memory is needed).

b) Fill in the table below:

Iteration	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Total number of holes	**1**	**4**	**13**	**40**	**121**	**364**	**1093**	**3280**

c) Do you see a **pattern** here? (Remember **exercise 1. (3) (4)** above) Use this pattern to **predict** the total number of holes in the 9th and in the 10th iteration.

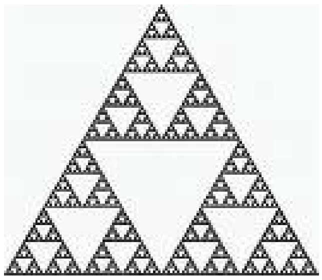
.

$1 + 3 = 4$ $\quad 4 + 3^2 = 13$ $\quad 13 + 3^3 = 40$ $\quad 40 + 3^4 = 121$ $\quad 121 + 3^5 = 364$

$364 + 3^6 = 1093$ $\quad 1093 + 3^7 = 3280$ $\quad 3280 + 3^8 = \underline{9841}$ $\quad 9841 + 3^9 = \underline{29\ 524}$

d) In your imagination, you could repeat this rule infinitely many times. The fate of this orbit is a shape known as

the Sierpiński triangle fractal.



2. Search for information on the Web and complete the following passage by filling in the blanks

Waclaw Sierpiński (**1882-1969**) was born in **Warsaw**, the capital of **Poland.**

He described his triangle in the year **1915**.

In 1920 he founded the mathematical journal **Fundamenta Matematicae**, still edited nowadays by the Polish Academy of Science.

3-Area of the Sierpiński triangle

Assume the **area** of the original triangle is **1 square unit.**

a) Look again at exercise 1 above and complete the grid below (round to two decimal places):

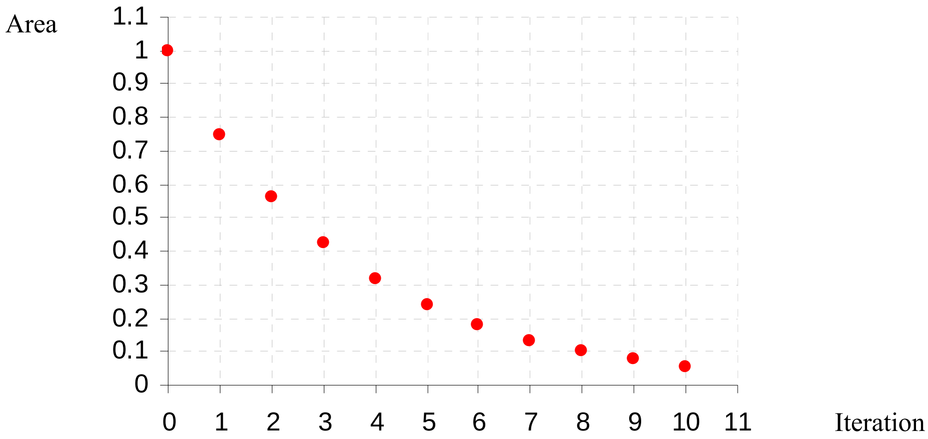
	Original triangle	First iteration	Second iteration	Third iteration	Fourth iteration
Area as a fraction (in square units)	**1**	$\frac{3}{4}$	$\left(\frac{3}{4}\right)^2 = \frac{9}{16}$	**$(3 / 4)^3 = 27 / 64$**	**$(3 / 4)^4 = 81 / 256$**
Area as a decimal (in square units)	**1**	**0.75**	**0.56**	**0.42**	**0.32**

b) Use a spreadsheet or calculator to compute the area of the next six iterations.

Enter the seed 1 into the first cell A1 and the formula = **A1*0.75**, into the second cell A2.

0.24, 0.18, 0.13, 0.10, 0.08, 0.06 rounded to 2 d.p.

c) Sketch a plot of these eleven points.



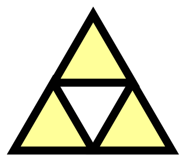
d) What can you explain about the area if we continue iterating?

The area is getting smaller and smaller because we are adding holes indefinitely.

e) Complete the sentence below:

The area of the Sierpiński triangle **is zero**.
Point out that any particularly iteration has area zero. The statement is about the fate of the orbit.

4-Perimeter of the Sierpiński triangle



- Remember the iteration rule:
"Remove the **interior** of the middle triangle".
So the **boundary** of the middle triangle **makes up part** of the **perimeter.**

- Assume the **side length** of the original triangle is **1 unit**.

a) Look again at exercise 1 above and complete the grid below:

	Original triangle	First iteration	Second iteration	Third iteration
Side length of each triangle remaining	**1**	$\frac{1}{2}$	$\frac{1}{4}$	**1 / 8**
Perimeter of each triangle remaining	**3 · 1 = 3**	$3 \cdot \frac{1}{2} = \frac{3}{2}$	**3 · 1 / 4 = 3 / 4**	**3 · 1 / 8 = 3 / 8**
Total number of triangles **remaining**	**1**	**3**	**9**	**27**

Perimeter	**$3 \cdot 1 = 3$**	**$(3/2) \cdot 3 = 9/2$**	**$(3/4) \cdot 9 = 27/4$**	**$(3/8) \cdot 27 = 81/8$**

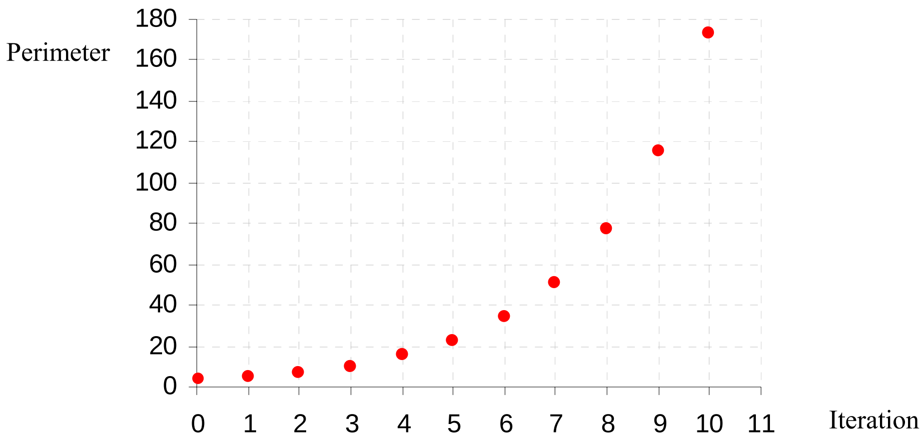
b) Do you see a **pattern** here? Use a spreadsheet or calculator to compute the perimeter of the next seven iterations.

$3^2 / 2 = 9/2$ **$3^3 / 2^2 = 27/4$** **$3^4 / 2^3 = 81/8$**

Enter the seed 3 into the first cell A1 and the formula = **A1*1.5**, into the second cell A2.

3, 4.5, 6.75, 10.13, 15.19, 22.78, 34.17, 51.26, 76.89, 115. 33, 172.96 rounded to 2 d.p.

c) Sketch a plot of these eleven points.



d) What's happening to the perimeter of the Sierpiński triangle?

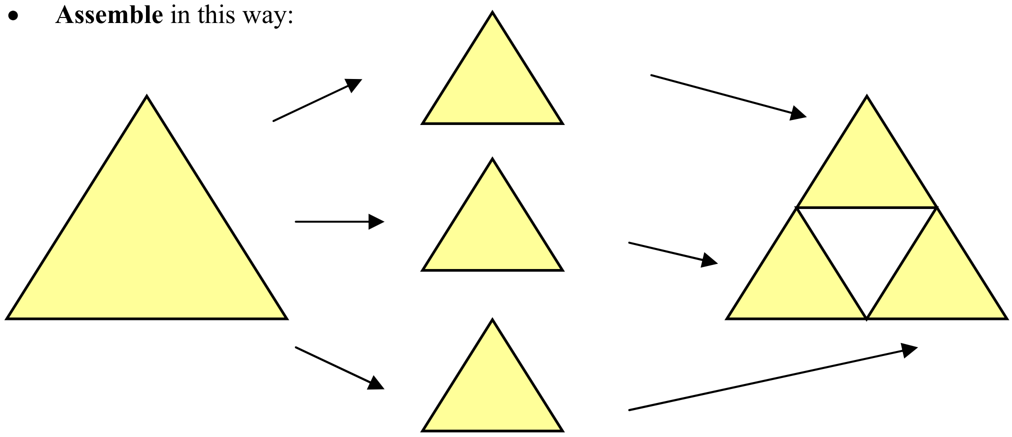
The perimeter is getting larger and larger. It tends to infinity.

The Sierpiński triangle has area zero and perimeter infinity!!!

5-Copies of copies

Use any computer **drawing program** in this exercise.
Start **again** with a filled equilateral triangle. **Everyone** in class should start with the **same large** triangle.
The iteration rule is:

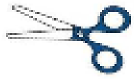
- Make three **copies** of the triangle.
- **Reduce** them by 50%.
- **Assemble** in this way:



Now draw the second, the third and the fourth iteration.
What shape will you get by continuing this process? **The Sierpiński triangle**

6-Poster presentation

- **Print** the fourth iteration and **cut out** the triangle.



- **Cooperate** with your classmates by making a **poster** of the **largest** Sierpiński triangle you can create:
  - **Assemble** all of the triangles by **matching** their vertices.

- Ask the following questions:
  - How many pupils are there in your class? Open answer
  - How many triangles did you use by making the poster? Open answer
  - How many copies do you need to be sure you get a Sierpiński triangle? List the possibilities and explain.

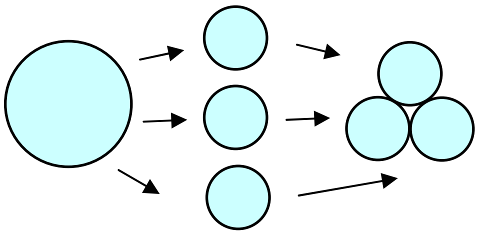
1, 3, 9, 27, 81.........We make 3 copies of each copy.

- If each original triangle has sides of length 15 cm, how many copies would you need to produce a Sierpiński triangle whose **base** is 60 cm long?

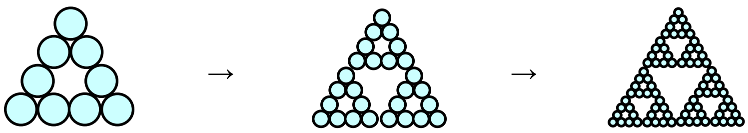
Sixty divide by fifteen is 4, so we need 4 copies on the base, 2 + 2 in the middle and 1 on the top. That is 9 copies.

7-Triplicating a circle

Start with a filled circle and follow the rule:
- Make three **copies** of the circle.
- **Reduce** them by 50%.
- **Assemble** in this way:



a) Sketch the next three iterations of this rule above.

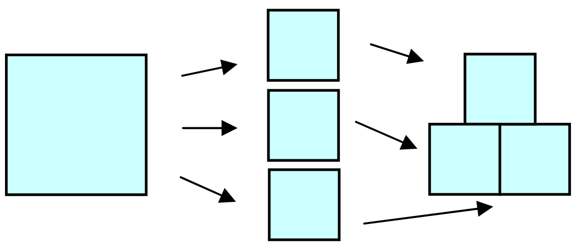


Second iteration **Third iteration** **Fourth iteration**

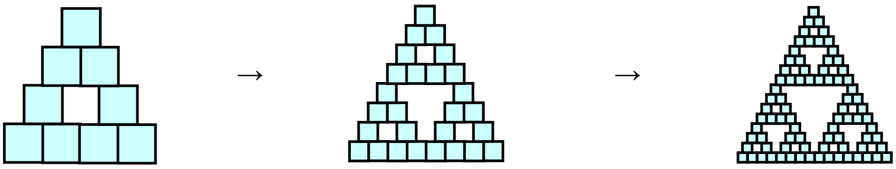
b) What figure will result if you continue to iterate this rule? **The Sierpiński triangle**

8-Triplicating a square

Start with a filled square and follow the rule:
- Make three **copies** of the square.
- **Reduce** them by 50%.
- **Assemble** in this way:



a) Sketch the next three iterations of this rule above.



Second iteration **Third iteration** **Fourth iteration**

b) What figure will result if you continue to iterate this rule? **The Sierpiński triangle**

9-Further exploration

Pick your own photo as your seed or any image you like and proceed with the rule as above. **Organise** your results as a poster and **describe** your work using the past "**I have** ...". **Include** your prediction of what you think will result if you continue to iterate this rule indefinitely.

I have chosen a photo of mine and I have reduced it.

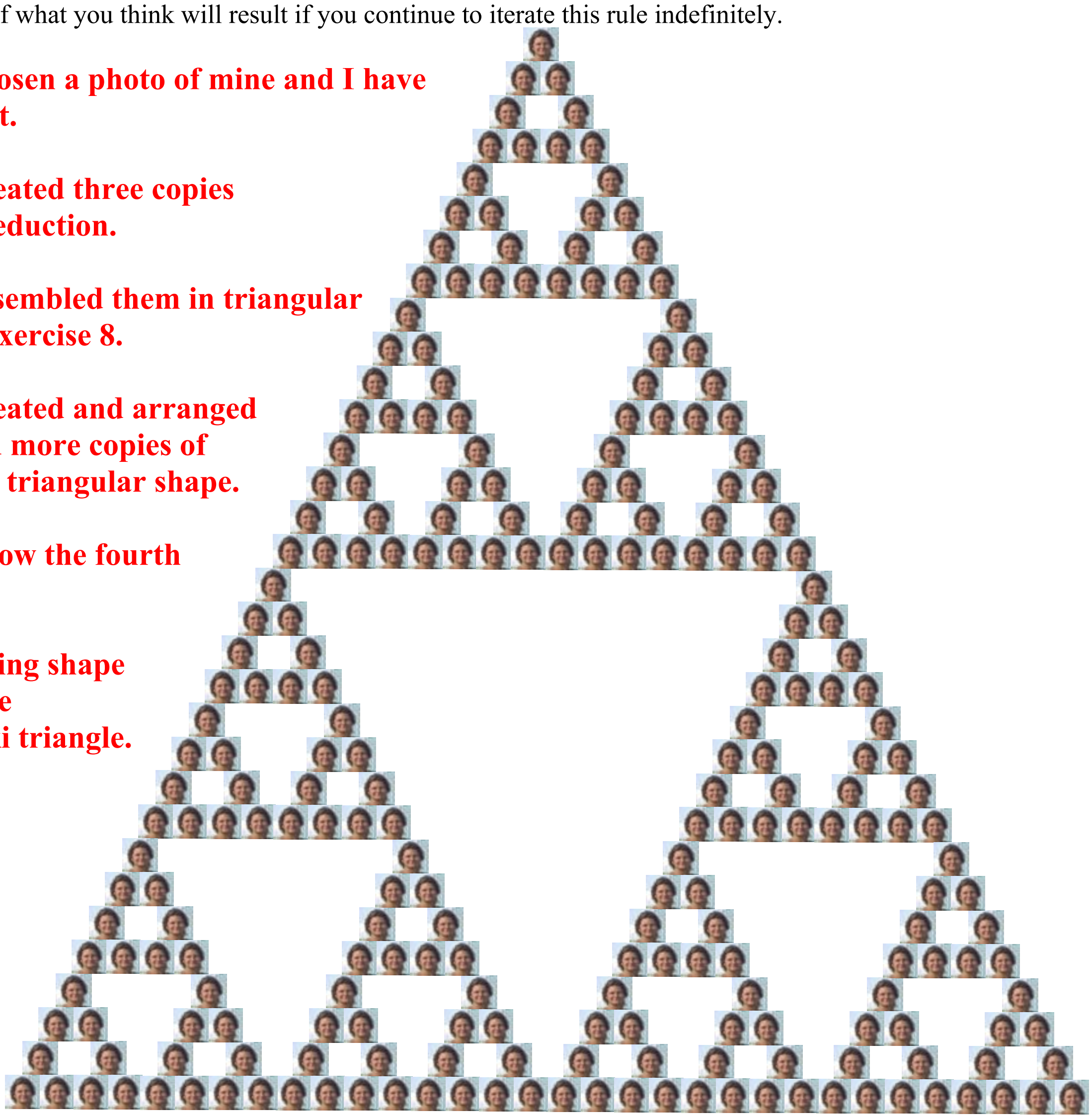
I have created three copies at 50% reduction.

I have assembled them in triangular form as exercise 8.

I have created and arranged more and more copies of this basic triangular shape.

Here I show the fourth iteration.

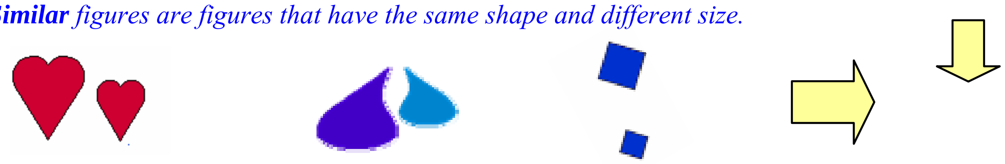
The limiting shape will be the Sierpiński triangle.



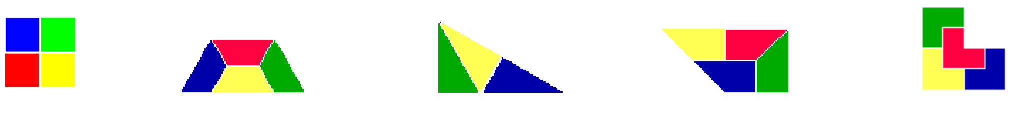
10-Self-similarity

Remember

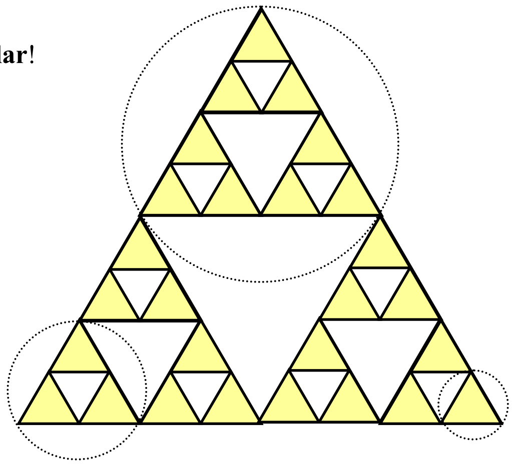
- ***Similar*** *figures are figures that have the same shape and different size.*



- *A* ***self-similar*** *figure is a figure made by small copies of itself.*



The Sierpiński triangle is **self-similar**!



- Take a **magnifying glass** to look more closely at the Sierpiński triangle:

Magnification is the amount by which a smaller self-similar image must be magnified to return to the original size. All magnifications are referring to linear measurements only.

1. How many copies do you see where **magnified by a factor of 2** yield the entire figure? **3**
2. And by a factor of 4? **9**
3. And by a factor of 8? **27**
4. Can you find a pattern here?

The Sierpiński triangle consists of three self-similar pieces that each have magnification factor of 2.
It also consists of 9 = 3^2 self-similar pieces that have magnification factor of 4 = 2^2.
And it also consists of 27 = 3^3 self-similar pieces with magnification factor 8 = 2^3.

- Go to: http://math.rice.edu/~lanius/fractals/selfsim.html and **zoom in**.
The triangle will look the same!

No matter how closely we look at this figure, we always find smaller and smaller copies of itself.

11-Pascal triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

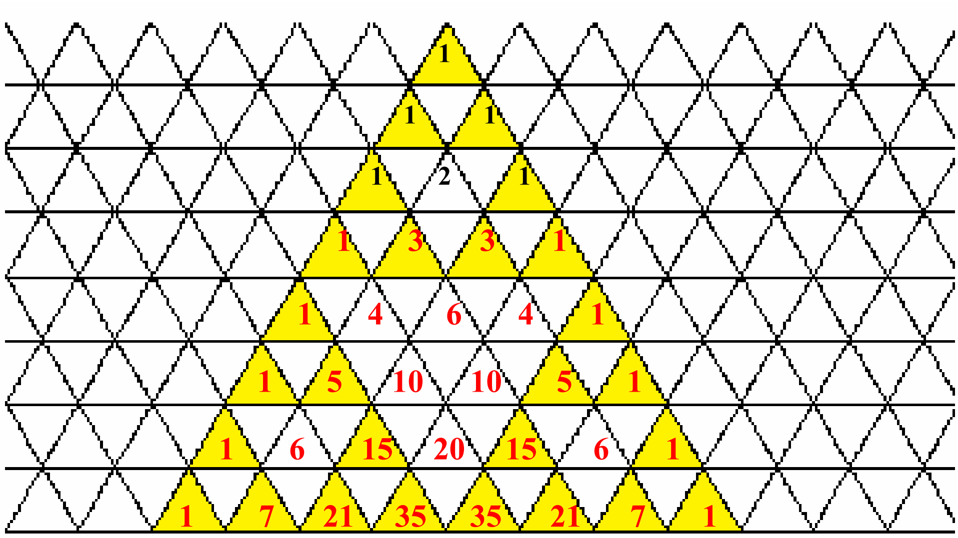
a) Do you see the pattern of how the numbers are placed in this triangle?
Fill in the next three rows.

Any row starts and finishes by 1. We get the middle values adding the number that can be seen above and to the left and the number above and to the right. For example, in the fifth row we have 1 + 3 = 4, 3 + 3 = 6, 3 + 1 = 4.

b) This number triangle was known in China and Persia in the eleventh century.
Later on the French mathematician Blaise Pascal, among others, studied its properties. In which **century** did Pascal live?
Pupils search the answer on the Internet: **Seventeenth century.**

c) Copy the Pascal triangle in the grid below.

d) **Shade** all the little **odd numbered** triangles in the Pascal triangle.

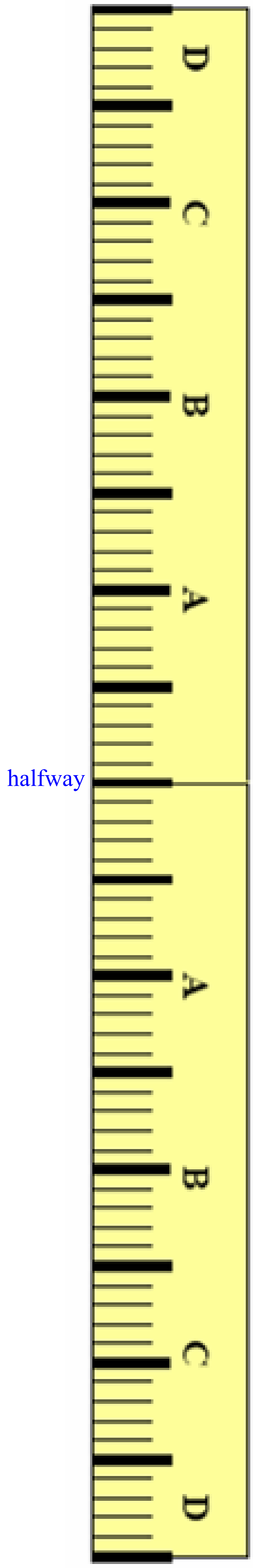


e) What figure did you get?
The third iteration of the Sierpiński triangle fractal.

12-The chaos game

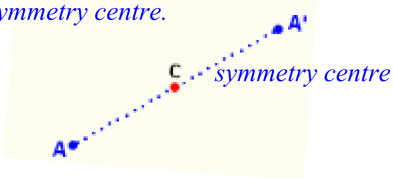
1. Number of players: in pairs.

2. Materials: worksheet, die, pencil and rubber, pen, ruler (better a half-ruler as showed on your right), transparency and marker pen.



Remember

Central symmetry is the reflection of an object through a point, called symmetry centre.



3. Half-ruler: It is handy to use a half-ruler for drawing. You can make your own half-ruler from cardboard.

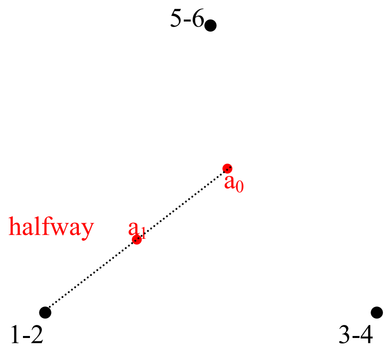
4. How to start playing?

- Place three points at the vertices of an equilateral triangle. Call them 1-2, 3-4 and 5-6.
- Choose **any** point in the interior of the triangle. Use a **pencil**. Call this point a_0.

5. How to play?

- First **roll the die** to determine a **vertex**.
- Then, using a pencil and a half-ruler, move to a new point which is **exactly halfway** between a_0 and the target vertex. Call this point a_1.

6. Example: if you roll a 2

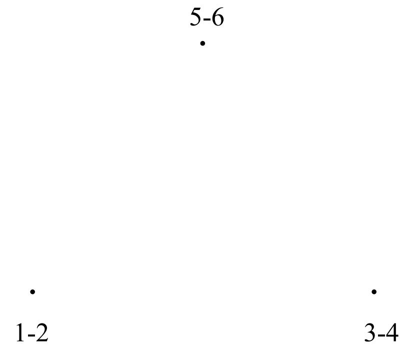


7. Playing the game

Play the chaos game with the given vertices below.

- Choose a point a_0 somewhere **inside** the triangle.
- Roll the die and plot the point a_1 using your half-ruler and **pencil**.
- Plot the points a_2, a_3, a_4, a_5 and a_6 determined by your die rolls.

Open answer.
The steps of the process will be random.
In order to make sure that you truly have a random process it helps if you erase the first points in the orbit.
That way, you will begin recording the orbit at a more random seed.



- **Switch to a pen** and plot a_7.
- **Erase** the first seven points a_0, a_1, a_2, a_3, a_4, a_5 and a_6.
- Plot the next ten points or more using a pen.
- Compare your results with those of your classmates.
- Copy your points onto a transparency. The rest of the class does the same.

Have two or three transparencies ready. Several pairs of pupils can record their points on the same transparency. Make sure the transparencies have the three vertices already on them to match up with worksheets.

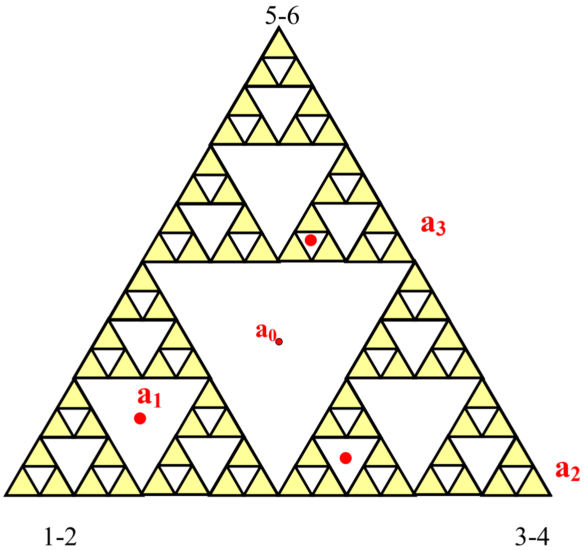
Lining all the transparencies up, do you see a pattern emerging? **It seems that the Sierpiński triangle is emerging.**

Pupils should realise at least that the middle section of the triangle is not being filled in. The access to the Web site below will help to see the emerging shape.

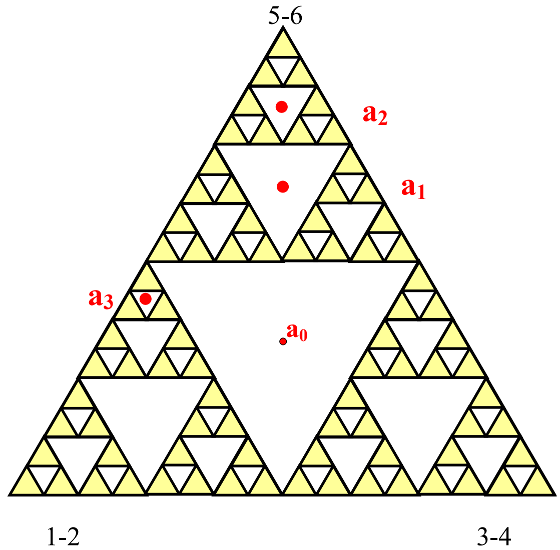
8. The game on the Web
Go to **http://www.jgiesen.de/ChaosSpiel/Spiel1English.html** and play the game plotting one point at a time.
Go to **http://www.jgiesen.de/ChaosSpiel/Spiel1000English.html** and play the game plotting one thousand points at a time.

9. Playing the game with the Sierpiński triangle showing
- **Choose** the point a_0 in the middle of the triangle and **plot** the points a_1, a_2 and a_3 in each picture below according to the three rolls of the die indicated.

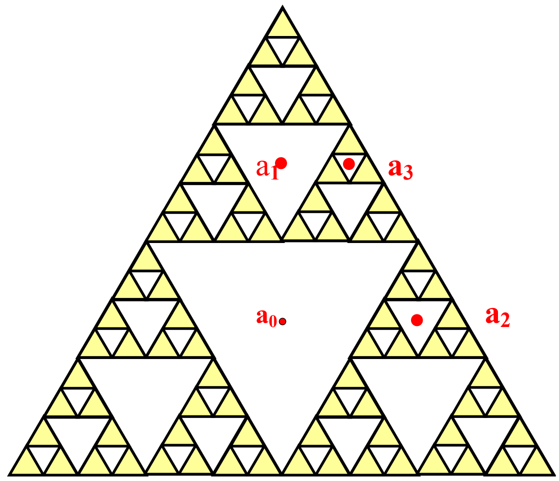
a) You roll 1, 4, 5.



b) You roll 6, 5, 2.



c) You roll 5, 3, 6.



- **Explain** how the points are distributed in the triangles.

a) The point a_1 is located halfway between a_0 and the vertex 1-2, so must land in one of the three smaller holes at the next level of the construction.

The point a_2 is located halfway between a_1 and the vertex 3-4, so must lie in one of the next smaller holes.

The point a_3 is located halfway between a_2 and the vertex 5-6, so must lie in one of the next smaller holes.

b) and c) Similar answers.

10. Conclusion

Argue with your partners and your teacher why after playing the chaos game thousands of times appears the image of the Sierpiński triangle.

In the chaos game the first points are erased so we don't see a_0, a_1, a_2 and a_3 in the final picture.

We see that each time we roll the die, our point moves into smaller and smaller missig triangles.

Although the point is still inside a microscopically small missing triangle, it is getting closer and closer to filled triangles. The point is slowly being attracted to the Sierpiński triangle.

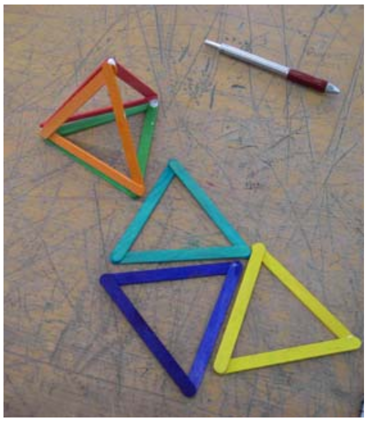
Something geometrically regular can emerge from a purely random game!!!

13-The Sierpiński Tetrahedron

Go to
http://www.toomates.net/llistes/geometria/figures_geometriques/fractals/tetraedre_sierpinski_bastonets_gelat/tetraedre_sierpinski_bastonets_gelat.htm

Look at the pictures and **cooperate** with all of your classmates by building your own Sierpiński Tetrahedron

Materials: ice-cream sticks and glue.



- **At stage 1** you put four small tetrahedra together.

How many small tetrahedra will you need at stage 2? **$4 \cdot 4 = 16$**

How many small tetrahedra will you need at stage 3? **$4 \cdot 4 \cdot 4 = 16 \cdot 4 = 64$**

How many small tetrahedra will you need at stage 4? **$4 \cdot 4 \cdot 4 \cdot 4 = 64 \cdot 4 = 256$**

Can you find a pattern here? **The number of tetrahedra at any given stage is always the number of tetrahedra in the previous stage, multiplied by four.**

How many small tetrahedra would you need at stage 10? **$4^{10} = 1\,048\,576$ tetrahedra**

- Suppose that the length of each stick is 10 cm.

How long will each edge be at stage 1? **$10 \cdot 2 = 20$ cm**
How long will each edge be at stage 2? **$10 \cdot 2 \cdot 2 = 40$ cm**
How long will each edge be at stage 3? **$10 \cdot 2 \cdot 2 \cdot 2 = 80$ cm**
How long will each edge be at stage 4? **$10 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 160$ cm**

Can you find a pattern here? **The edge length at any given stage is always the edge length in the previous stage, multiplied by two.**

How long will each edge be at stage 10? **$10 \cdot 2^{10} = 10\,240$ cm**

The Sierpiński Tetrahedron is **self-similar**!