LESSON PLAN 4. Koch curve and Snowflake

Aim: To introduce pupils to one of the most popular and well known fractal. The two ways to generate fractals geometrically, by “removals” and “copies of copies”, are revisited. Pupils should begin to develop an informal concept of what fractals are.

Teaching objectives:

Content

- Geometric iteration rule by removals
- Copies of copies method
- Areas and perimeters
- Numerical and geometric patterns
- Geometric transformations. Friezes
- Self-similarity

Communication

- Predicting: “will be...”, “would be ...”,
- Comparing: “getting smaller ...”, “larger than...”, “magnified by a factor of ...“, “increasing by a factor of ...”
- Describing your work: “I have chosen ...”,...
- Arguing with partners predictions and conclusions
- Translating visual information into verbal descriptions
- Checking calculations in pairs
- Sharing opinions to draw a frieze
- Reading summary sentences to other groups
- Writing a script for an oral presentation

Cognition

- Identifying similar shapes
- Predicting eventual outcomes
- Calculating parts of a unit whole, perimeters and areas
- Creating figures through drawing programs
- Creating spreadsheets to display data
- Plotting a graph to illustrate selected information
- Writing summary reports using key vocabulary
- Arguing and checking answers
- Using Internet resources to confirm predictions

Context

- Zooms of the size of images
- Computer graphics and visual art
- Frieze decoration
- History of Mathematics

Outcomes: At the end of the lesson, pupils will be able to

- Distinguish between two similar figures and a self-similar figure
- Identify numerical and geometric patterns
- Translate visual information to verbal descriptions
- Use drawing programs to generate further iterative outputs
- Predict what will happen if we continue iterating
- Confirm predictions using the Internet resources
- Be confident in geometric English language
- Develop a sense of responsibility when working in a cooperative group
- Recognise the importance of arguing in a team
- Know how to behave in a discussion
- Present their work to the rest of the class

Key words: geometric iteration rule, Koch curve, Koch Snowflake, self-similarity, frieze.

Prerequisites: Pupils need to be familiar with straight line segments, equilateral triangles, polygons, whole powers and fractions as a part of a whole.

Curriculum connections:

- Calculation of perimeters and areas.
- Computation with fractions in a geometric context.
- Triangle's geometry.
- Practise with recognizing and describing numerical and geometric patterns.
- Geometric transformations. Similarity.
- Cartesian coordinates and graph plotting.
- Use of calculator or computer when needed.

Timing: Six 55-minute class periods.

Tasks planned:

First session

- Provide pupils with worksheets 1-*Removals* and 2-*Koch curve*. Let them work on their own. Only circulate among them to be sure they are headed in the right direction.
- The indicated Web site is more interesting than others because the quick increasing of the length of the curve can easily be observed. Point out this fact.
- Plenary: sharing and discussing solutions.
- Homework: Ask to finish accurately the exercise.

Second session

- Pupils work through 3-*Length of the Koch curve*. The first two columns of the grid are known from the previous worksheet.
- Plenary: correction of exercise 3 and comments on the conclusion.

Third session

- Pupils work through exercise 4-*Copies of copies* by using a computer drawing program.
- The entire class cooperates by making ornamental friezes for decoration.
- Homework: Ask pupils to explore different possibilities.

Fourth session

- Each pupil presents to the rest of the class his/her home exploration.
- Final conclusion: teacher could take this opportunity to explain that there are only seven possible frieze patterns.
  Go to: http://mathcentral.uregina.ca/RR/database/RR.09.01/mcdonald1/
- Homework: exercise 6-*Self-similarity*.

Fifth session

- Homework revision: Ask questions to be sure that pupils understand what self-similarity is.
- Pupils work through exercise 7-*The Koch Snowflake* and 8-*Perimeter of the Koch Snowflake.*
- Homework: Ask pupils to finish exercise 8.

Sixth session

- Homework revision.
- Pupils work through exercise 8-*Area of the Koch Snowflake.*
- Plenary: Final conclusions about perimeter and area of the Koch Snowflake.

Resources

- For class activities and homework: the pupils' worksheets and access to a computer drawing program.
- For drawings and making friezes: ruler, pencil and rubber, coloured pencils, cardboard, scissors and glue, pen, markers ...
- For teaching instructions and answer keys to each exercise: the teachers' worksheets, mainly borrowed from the book *Fractals* by J. Choate, R.L. Devaney, A. Foster, listed in the bibliography.

Assessment

- The teacher will take notes of pupils' progress every period.
- Good presentation in worksheets and active participation in the classroom is essential.
- Friezes will be individually marked depending on participation, contribution to the group work and oral presentation.
- Homework will be suitably marked.
- Pupils who talk in English will be given extra marks.

Evaluation

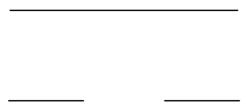
TEACHER'S NOTES AND ANSWERS

Answers to exercises

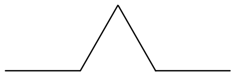
1-Removals

Here is a geometric iteration:

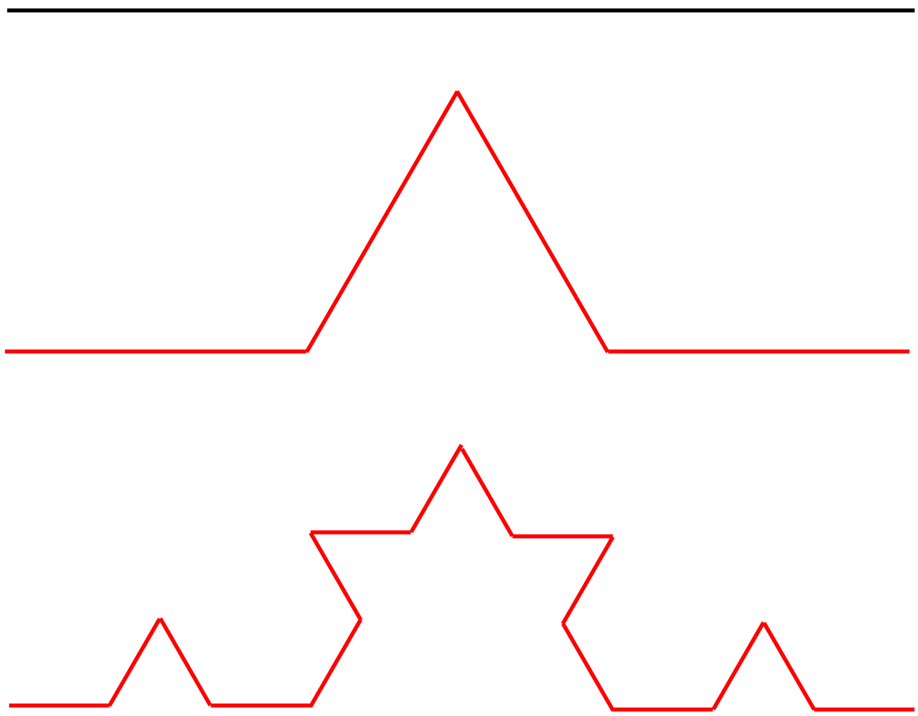
Seed: a straight line segment
Iteration rule:



- Divide the segment into three equal parts
- **Remove** the middle third
- Replace it with two segments the same length as the section you removed.



1. Start with the straight line segment below and sketch the first and the second iteration.



2. - How many segments does the first iteration have? **4 segments**

   - How many segments does the second iteration have? **4 times 4 = 4^2 = 16 segments**

   - How many segments **will** the third iteration have? **4 times 16 = 4^3 = 64 segments**

   - Do you see a **pattern** here? Explain.
   From every segment we get 4 smaller segments in the next iteration. So we have to multiply by 4 at each stage.

2-Koch curve

1. Go to: http://www.arcytech.org/java/fractals/koch.shtml

This applet is implemented in such a way that you can see the fractal as it gets drawn.

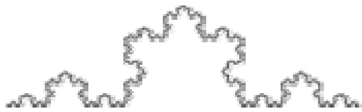
a) Play the game and move from one iteration to the next.
Stop at the 11th iteration (it takes a long time to draw and a good memory is needed).

b) How many segments does the eleventh iteration have?

We begin with 1 segment and we have to multiply eleven times by 4, that is 4^{11} = 4 194 304 segments.

Pupils should notice that it takes a long time to draw more than 4 million small segments.

c) In your imagination, you could repeat this rule infinitely many times.
The orbit of this iteration rule tends to famous fractal known as

the Koch curve 

2. Search for information on the Web and complete the following passage by filling in the blanks

Niels Fabian Helge Koch (**1870** - **1924**) was born in **Stockholm**, the capital of **Sweden**.
He described this curve in the year **1904**.

3. Assume the **length** of the original straight line segment is **1 unit**.

-What **fraction** is the length of each segment at the first iteration?
1 / 3

-What fraction is the length of each segment at the second iteration?

1/3 of 1/3 = 1/3 times 1/3 = 1/9

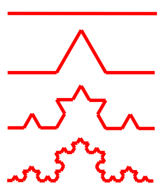
-What fraction **will be** the length of each segment at the third iteration?

1/3 of 1/9 = 1/3 times 1/9 = 1/27

- Do you see a **pattern** here? Use this pattern to predict what fraction **would be** the length of each segment at the fourth iteration.

$(1/3)^2 = 1/9$ **$(1/3)^3 = 1/27$** **$\underline{(1/3)^4 = 1/81}$**

3-Length of the Koch curve



Assume the **length** of the original straight line segment is **1 unit**.

1. Complete the grid below (round to two decimal places):

	Number of segments	Length of each segment	Total length (as a fraction)	Total length (as a decimal)
Original segment	**1**	**1**	**1**	**1**
First iteration	**4**	$\frac{1}{3}$	**$4 \cdot (1/3) = 4/3$**	**1.33**
Second iteration	**$4^2 = 16$**	**$(1/3)^2 = 1/9$**	**$4^2 \cdot (1/3)^2 = 16/9$**	**1.78**
Third iteration	**$4^3 = 64$**	**$(1/3)^3 = 1/27$**	**$4^3 \cdot (1/3)^3 = 64/27$**	**2.37**
Fourth iteration	**$4^4 = 256$**	**$(1/3)^4 = 1/81$**	**$4^4 \cdot (1/3)^4 = (4/3)^4$**	**3.16**

2. Complete the sentences below:

- The total length of each iteration is **4/3** times the total length of the iteration before.

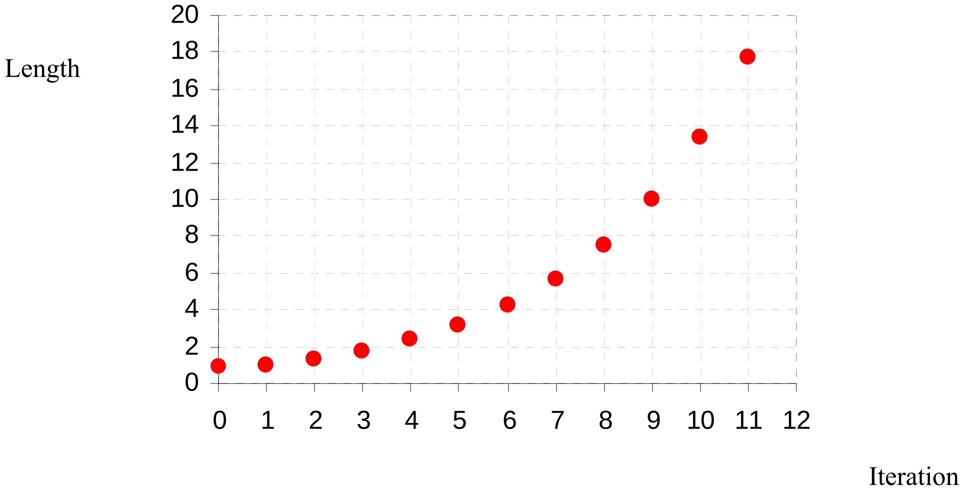
- The total length is **increasing** by a scale factor of **4/3**.

- The total length of the fifth iteration is **$(4/3)^5 \approx 4.21$** units.

3. Use a spreadsheet or calculator to compute the total length of the next seven iterations.

Enter the seed 1 into the first cell A1 and the formula = **A1*4/3**, into the second cell A2.
4.21, 5.62, 7.49, 9.99, 13.32, 17.76 units (rounded to 2 d.p.)

4. Sketch a plot of these twelve points.



5. How many iterations would it take to obtain a total length of 100 units, or as close to 100 as you can get?

At the sixteenth iteration the length is 99.77 units (rounded to two d.p.)

6. How many iterations would you have to draw to guarantee that the length is **larger** than 200 units?

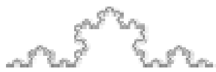
We have to draw 19 iterations, because at the eighteenth iteration the length is approximately 177.38 units and at the nineteenth iteration the length is 236.50 units.

7. Compare with your partner the answers above to number 5 and number 6. What can you explain about it?

We need 16 iterations to get a length of 100 units, but with only 3 more iterations the length is growing more than 100 units.

8. Write down a formula to calculate the length of the curve at the thousandth iteration.

$$(4 / 3)^{1\,000}$$



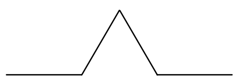
9. What can you explain about the length if we continue iterating?

The length keeps increasing quite quickly. The Koch curve is infinitely long!!

4-Copies of copies

Use any computer **drawing program** in this exercise.
Start **again** with a straight line segment. **Everyone** in class should start with the **same size** segment.
The iteration rule is:

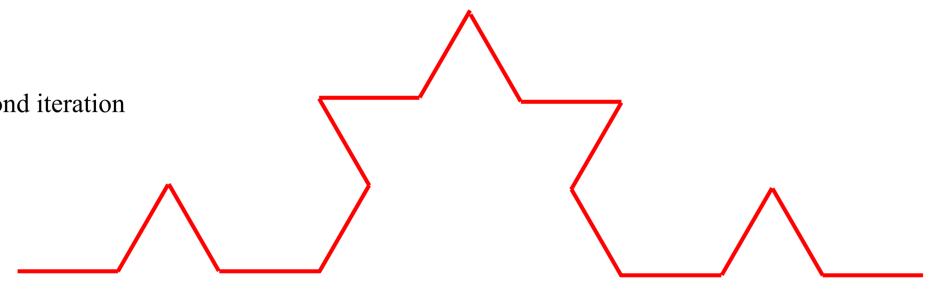
- Make four **copies** of the segment, each **reduced** to one-third the original size.
- **Assemble** in this way:



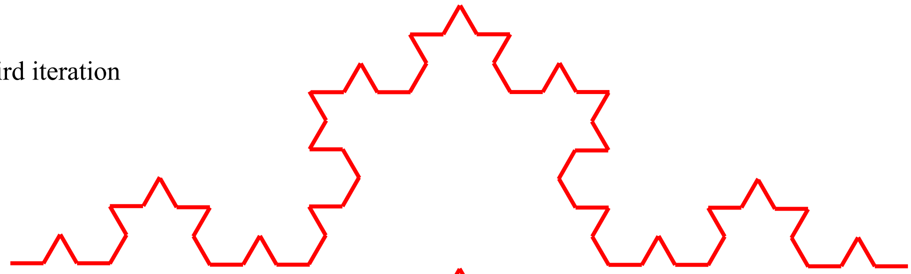
1. From left to right the second segment is **rotated counter clockwise** and the third one is rotated **clockwise.**
   What is the angle of rotation? **60º**

2. Now draw the second, the third and the fourth iteration.

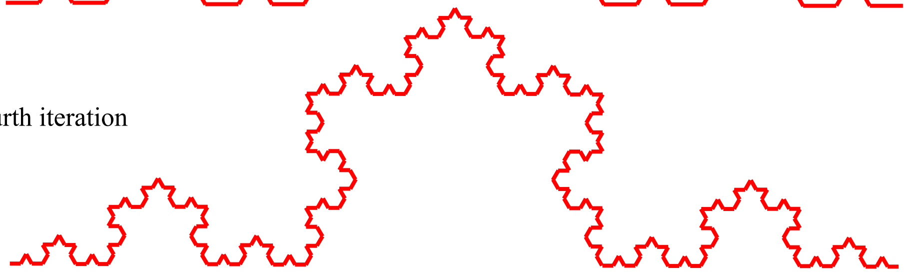
Second iteration



Third iteration



Fourth iteration



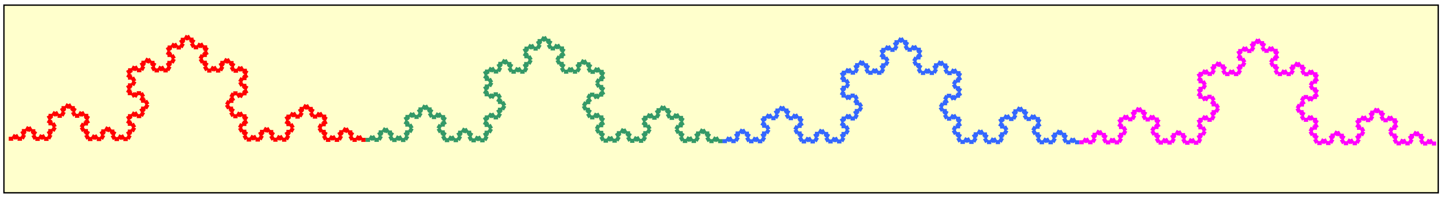
3. Go to: http://www.xtec.cat/~dobrador/abeam/koch.swf
   What shape will you get by continuing this process? **The Koch curve**

5-Frieze presentation

1. **Print** the fourth iteration.

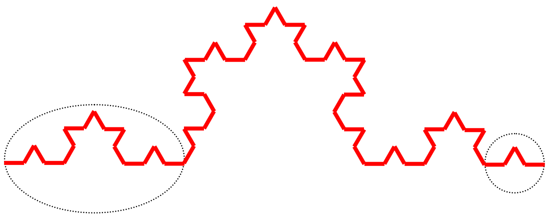
2. **Cooperate** with your classmates by making an ornamental **frieze** to decorate your classroom.

Here is an example, but other possibilities could be explored.



6-Self-similarity

The Koch curve is **self-similar**!



- Take a **magnifying glass** to look more closely at the Koch curve:

Magnification is the amount by which a smaller self-similar image must be magnified to return to the original size. All magnifications are referring to linear measurements only.

a) How many copies do you see where **magnified by a factor of 3** yield the entire figure? **4 copies**

b) And by a factor of 9? **16 copies**

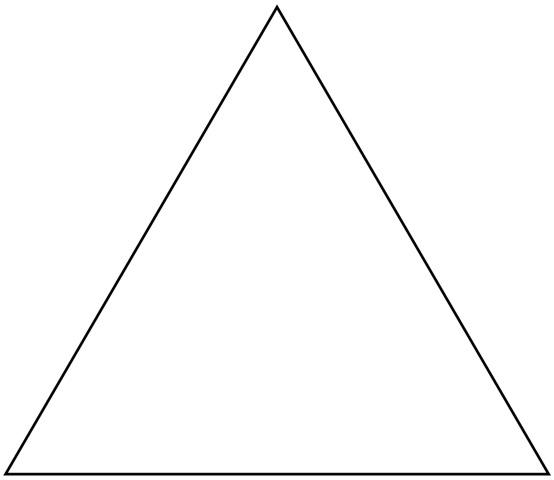
- Go to**:** **http://www.jimloy.com/fractals/koch.htm** and look at the movie **Zooming in on the Koch Curve**.
The curve looks the same! **Miniature copies of itself can be found all along the curve!**

7-The Koch Snowflake

Start with an equilateral triangle.
Apply the iteration rule:

- Divide **each side** of the triangle into three equal parts.
- **Remove** the middle third.
- Replace it with two segments the same length as the section you removed.

1. Sketch the first and the second iteration.



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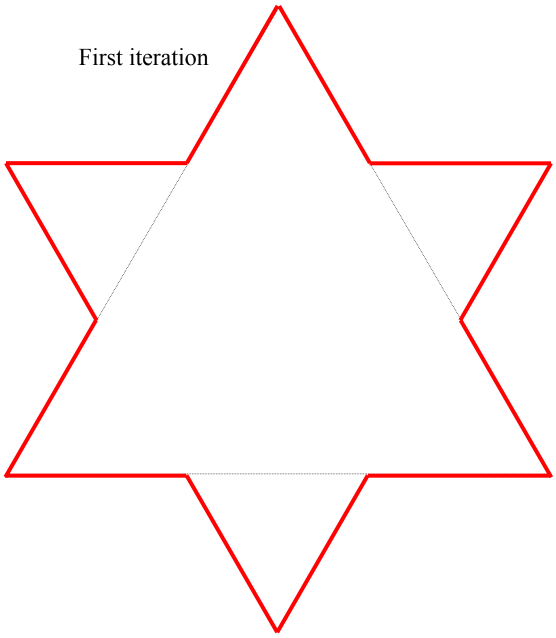
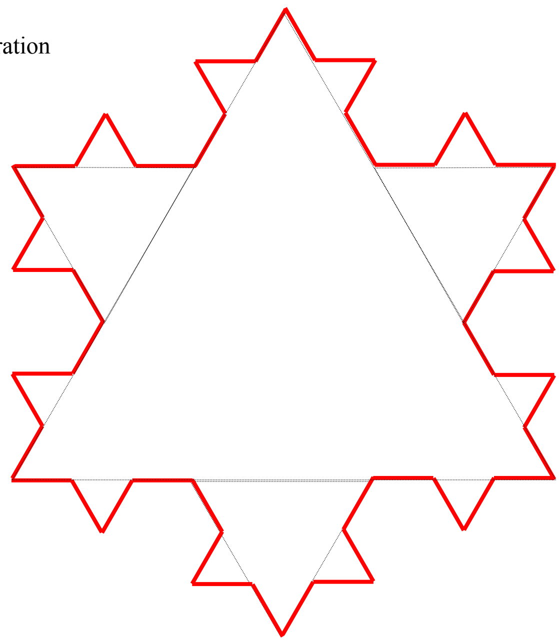
First iteration



Second iteration



2. - How many sides does the first **polygon** have? **4 times 3 = 12 sides**
   - How many sides does the second polygon have? **4 times 12 = 48 sides**
   - How many sides **will** the third iteration have? **4 times 48 = 192 sides**
   - Do you see a **pattern** here? Explain.

From every side we get 4 smaller sides in the next iteration. So we have to multiply by 4 at each stage.

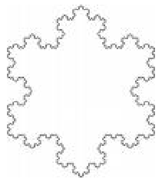
3. Go to: http://math.rice.edu/~lanius/frac/koch/koch.html

a) Play the game and move from one iteration to the next.

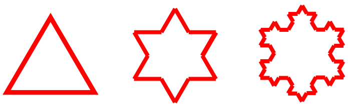
b) How many sides does the sixth iteration have? $\mathbf{4^6 \cdot 3 = 12\,288}$ **sides**

c) In your imagination, you could repeat this rule infinitely many times.
The orbit of this iteration rule tends to famous fractal known as

the Koch Snowflake curve



8-Perimeter of the Koch Snowflake



Assume the **perimeter** of the original triangle is **3 units**.

1. Complete the grid below (round to two decimal places):

	Number of sides	Length of each side	Perimeter (as a fraction)	Perimeter (as a decimal)
Original triangle	**3**	**1**	$\mathbf{3 \cdot 1 = 3}$	**3**
First iteration	$\mathbf{4 \cdot 3 = 12}$	$\frac{1}{3}$	$\mathbf{12 \cdot (1/3) = 12/3 = 4}$	**4**
Second iteration	$\mathbf{4^2 \cdot 3 = 48}$	$\mathbf{(1/3)^2 = 1/9}$	$\mathbf{4^2 \cdot 3 \cdot (1/3)^2 = 4^2/3 = 16/3}$	**5.33**
Third iteration	$\mathbf{4^3 \cdot 3 = 192}$	$\mathbf{(1/3)^3 = 1/27}$	$\mathbf{4^3 \cdot 3 \cdot (1/3)^3 = 4^3/3^2 = 64/9}$	**7.11**
Fourth iteration	$\mathbf{4^4 \cdot 3 = 768}$	$\mathbf{(1/3)^4 = 1/81}$	$\mathbf{4^4 \cdot 3 \cdot (1/3)^4 = 4^4/3^3 = 256/27}$	**9.48**

2. Complete the sentences below:

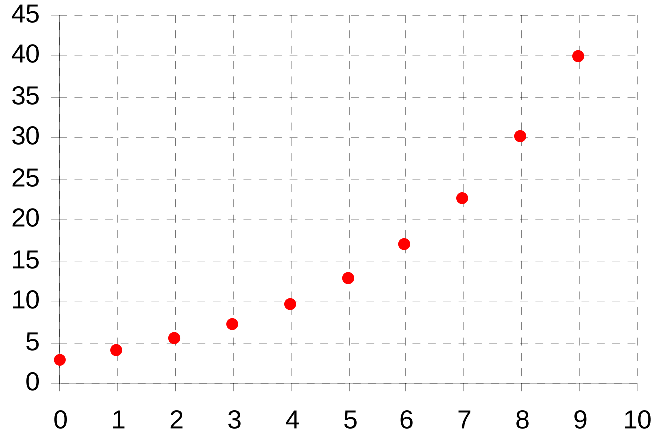
- The perimeter of each figure is **4/3** times the perimeter of the figure before.
- The perimeter is **increasing** by a scale factor of **4/3**
- The perimeter of the fifth iteration is $\mathbf{4^5/3^4 \approx 12.64}$ units.

3. Use a spreadsheet or calculator to compute the perimeter of the next five iterations.

12.64, 16.86, 22.47, 29.97, 39.95 units (rounded to 2 d.p.)

4. Sketch a plot of these ten points.

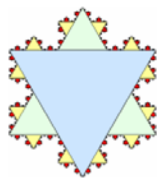
Perimeter



Iteration

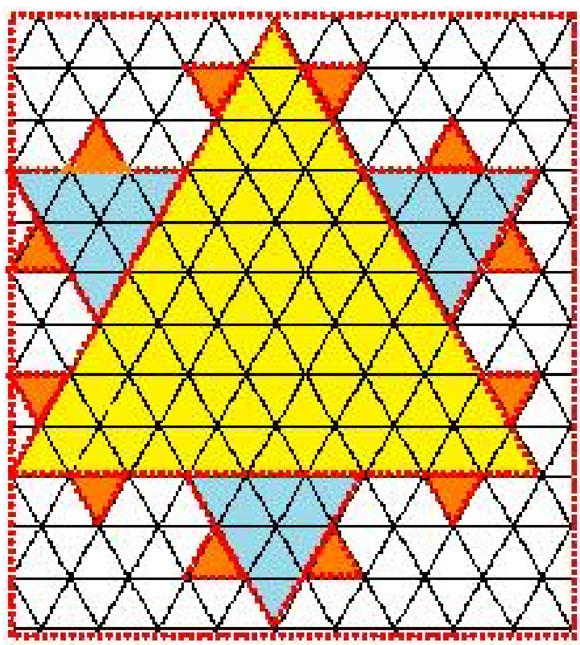
5. What can you explain about the perimeter if we continue iterating?

The perimeter increases by a factor greater than 1, so the Koch Snowflake is infinitely long!

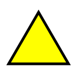


8-Area of the Koch Snowflake

1. Use coloured pencils to sketch below how the second iteration looks like.



2. Let's measure the area using the small triangles of our grid as units.

1 area unit = 

	Number of triangles added	Area of each triangle added	Amount of area added	Total area
Original triangle				**81**
First iteration	**3**	**9**	**27**	**108**
Second iteration	**12**	**1**	**12**	**120**

3. Complete the sentences below:

- The number of triangles added is **4** times the previous one.

- The area of each triangle added is **1/9** the previous one.

- The amount of area added is **increasing** by a scale factor of **4/9**

4. Predict **with your partner** the next five rows of the table:

Third iteration	$3 \cdot 4^2 = 48$	1/9	5.33	125.33
Fourth iteration	$3 \cdot 4^3 = 192$	$1/9^2$	2.37	127.7
Fifth iteration	$3 \cdot 4^4 = 768$	$1/9^3$	1.05	128.75
Sixth iteration	$3 \cdot 4^5 = 3\ 072$	$1/9^4$	0.47	129.22
Seventh iteration	$3 \cdot 4^6 = 12\ 288$	$1/9^5$	0.21	129.43

5. What is remarkable about the area and the perimeter of this shape?

The area is increasing but more and more slowly. It tends to some number because is a finite area.
The Koch snowflake has infinite perimeter but encloses finite area!

In fact, the Koch snowflake has 8/5 of the area of the original triangle, what are 129.6 units.

6. Go to: http://math.rice.edu/~lanius/frac/koch3.html and check your answers.