LESSON PLAN 5. Fractal Dimension

Aim: To revisit the traditional concept of dimension and to introduce pupils to a new type of dimension called fractal dimension. Pupils should gain a feeling for fractal dimension as a measure of the thickness or roughness of the fractal.

Teaching objectives:

Content
- Dimension: length, width and height
- Copies of copies
- Self-similarity
- Fractal dimension
- The Ruler method
- The Box Counting method

Communication
- Predicting: "will be...", "would be ...",
- Comparing: "magnified by a factor of ..."
- Describing your work: "I have chosen ...",...
- Arguing with partners predictions and conclusions
- Translating visual information into verbal descriptions
- Checking calculations in pairs
- Reading summary sentences to other groups

Cognition
- Identifying length, width and height
- Identifying similar shapes
- Developing intuition for dimension
- Measuring dimensions through iterative procedures
- Predicting eventual outcomes
- Creating figures through drawing programs
- Linearising data
- Plotting a graph to illustrate selected information
- Writing summary reports using key vocabulary
- Arguing and checking answers

Context
- Zooms of the size of images
- Computer graphics and visual art

Outcomes: At the end of the lesson, pupils will be able to
- Measure dimensions through iterative procedures
- Identify numerical and geometric patterns
- Translate visual information to verbal descriptions
- Use drawing programs to generate further iterative outputs
- Predict what will happen if we continue iterating
- Be confident in geometric English language

- Develop a sense of responsibility when working in a cooperative group
- Recognise the importance of arguing in a team
- Know how to behave in a discussion

Key words: Sierpiński triangle, Koch curve, self-similarity, fractal dimension, Ruler method, Box Counting method.

Prerequisites: Pupils need to be familiar with the Sierpiński triangle and the Koch curve from the previous units and with exponents.

Curriculum connections:

- Positive and negative powers of 10.
- Computation with exponents in a geometric context.
- Dimension.
- Practise with recognizing and describing numerical and geometric patterns.
- Geometric transformations. Similarity.
- Cartesian coordinates and graph plotting.
- Linear functions.
- Use of calculator or computer when needed.

Timing: Four 55-minute class periods.

Tasks planned:

First session

- Provide pupils with worksheets 1-*Dimension* and 2-*The Sierpiński triangle.* Be sure that pupils are entering into the concept of dimension.
- Plenary: explanation and comments on the fractal dimension of the Sierpiński triangle.
- Homework: exercise 3-*The Sierpiński Tetrahedron.*

Second session

- Homework revision: Sharing opinions about fractional dimension.
- Pupils work through 4-*The Koch curve.*
- Plenary: correction of exercise 4 and comments on the conclusion.

Third session

- Pupils work through exercise 5- *Measuring the Koch curve*. Let them experiment with the log key in their calculators without knowing logarithms. Maybe negative exponents would need an explanation.
  You can take this opportunity to talk about direct proportion and slope of a straight line.
- Plenary: correction of exercise 5 and comments on the conclusion.

Fourth session

- Pupils work through exercise 6- *The Box Counting method.*
- Plenary: Sharing results and final conclusions about the fractal dimension of the Koch curve.

Resources

- For class activities and homework: the pupils' worksheets and access to a computer drawing program.
- For *The Ruler and the Box Counting method*s: a strip of paper, ruler and compasses, pencil and rubber, coloured pencils, scissors , pen, markers ...
- For teaching instructions and answer keys to each exercise: the teachers' worksheets, mainly borrowed from the book *Fractals* by J. Choate, R.L. Devaney, A. Foster, listed in the bibliography.

Assessment

- The teacher will take notes of pupils' progress every period.
- Good presentation in worksheets and active participation in the classroom is essential.
- Homework will be suitably marked.
- Pupils who talk in English will be given extra marks.

Evaluation

TEACHER'S NOTES AND ANSWERS

Answers to exercises

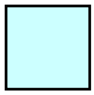
1-Dimension

Complete the sentences with the words in the box.

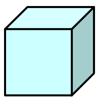
one	two	three	length	width	depth

1. A straight line segment has **one** dimension: **length**

2. The interior of a square has **two** dimensions: **length** and **width**



3. The interior of a cube has **three** dimensions: **length, width** and **depth**



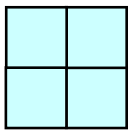
2-The Sierpiński triangle

1. Take the straight line segment below and double its length.

How many copies of the original segment do you get? **Two copies**

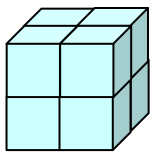
2. Take the square and double its length and width

How many copies of the original square do you get? **Four copies**



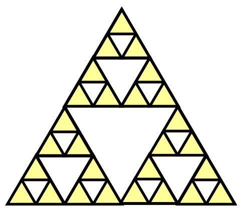
3. Take the cube and double its length, width and height.

How many copies of the original cube do you get? **Eight copies**



4. Take the Sierpiński triangle and double the length of his sides

How many copies of the original triangle do you get? **Three copies**



5. Organise your information into the table

Figure	Dimension	Number of copies	Number of copies as a power of two
straight line segment	**1**	**2**	2^1
square	**2**	**4**	2^2
cube	**3**	**8**	2^3
Sierpiński triangle	**?**	**3**	$2^?$

Do you see a pattern here? **When we double the sides and get a self-similar figure, we write the number of copies as a power of 2 and the exponent is the dimension**.

Experiment with the **exponent key** of your calculator to find out the fractal dimension of the Sierpiński triangle (**correct** to two decimal places).

The fractal dimension of the Sierpiński triangle will be the exponent *d* such that $2^d = 3$.

As $2^1 = 2$ and $2^2 = 4$, the dimension *d* will be a number between 1 and 2.

Let's try

$2^{1.5} \approx 2.828427124$ $2^{1.6} \approx 3.031433133$ $2^{1.57} \approx 2.969047141$ $2^{1.58} \approx 2.989698497$ $2^{1.59} \approx 3.010493494$

The fractal dimension of the Sierpiński triangle will be approximately 1.58, correct to two decimal places.

3-The Sierpiński Tetrahedron

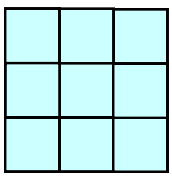
- Take the Sierpiński tetrahedron and double the edge length
- How many copies of the original tetrahedron do you get? **Four copies**
- Write down the number of copies as a power of two. 2^2
- What is the fractal dimension of the Sierpiński tetrahedron? **2, a whole number!**

Fractals don't need to have fractional dimension. Indeed, a fractal object must have fractal dimension greater than his topological dimension.

4- The Koch curve

1. Take the straight line segment below and triple its length.

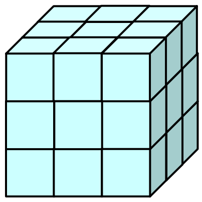
How many copies of the original segment do you get? **Three copies**



2. Take the square and triple its length and width

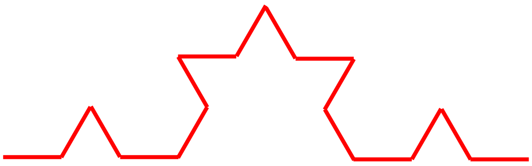
How many copies of the original square do you get? **Nine copies**

3. Take the cube and triple its length, width and height.



How many copies of the original cube do you get? **Twenty seven copies**

4. Take the Koch curve and magnify it by a factor of 3



How many copies of the original curve do you get? **Four copies**

5. Organise your information into the table

Figure	Dimension	Number of copies	Number of copies as a power of three
straight line segment	**1**	**3**	3^1
square	**2**	**9**	3^2
cube	**3**	**27**	3^3
Koch curve	**?**	**4**	$3^?$

Do you see a pattern here? **When we magnify by a factor of 3 and get a self-similar figure, we write the number of copies as a power of 3 and the exponent is the dimension**.

Experiment with the **exponent key** of your calculator to find out the fractal dimension of the Koch curve (**correct** to two decimal places).

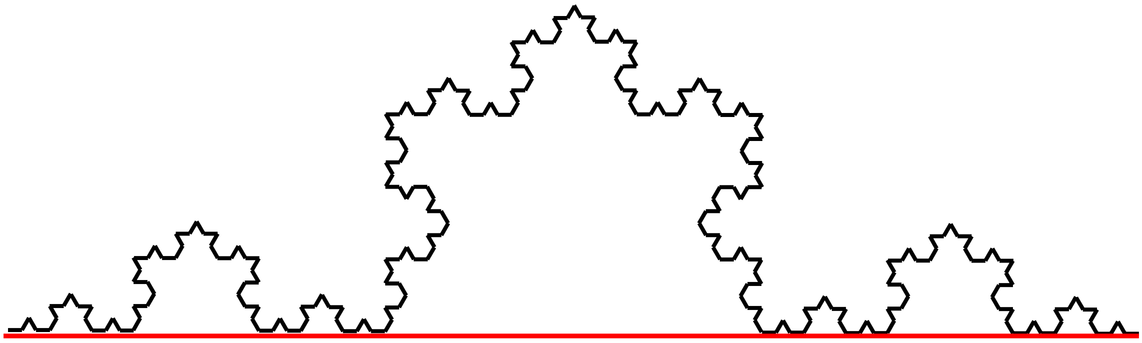
The fractal dimension of the Koch curve will be the exponent d such that $3^d = 4$.
As $3^1 = 3$ and $3^2 = 9$, the dimension d will be a number between 1 and 2.
Let's try
$3^{1.2} \approx 3.737192818$ $3^{1.3} \approx 4.171167510$ $3^{1.25} \approx 3.948222038$ $3^{1.26} \approx 3.991836831$
$3^{1.27} \approx 4.035933423$
The fractal dimension of the Koch curve will be approximately 1.26, correct to two decimal places.

6. Compare the fractal dimension of the Koch curve to the fractal dimension of the Sierpiński triangle.

The fractal dimension of the Koch curve is smaller than the fractal dimension of the Sierpiński triangle. This is just as we could expect, as the Koch curve appears to be less thick than the Sierpiński triangle.

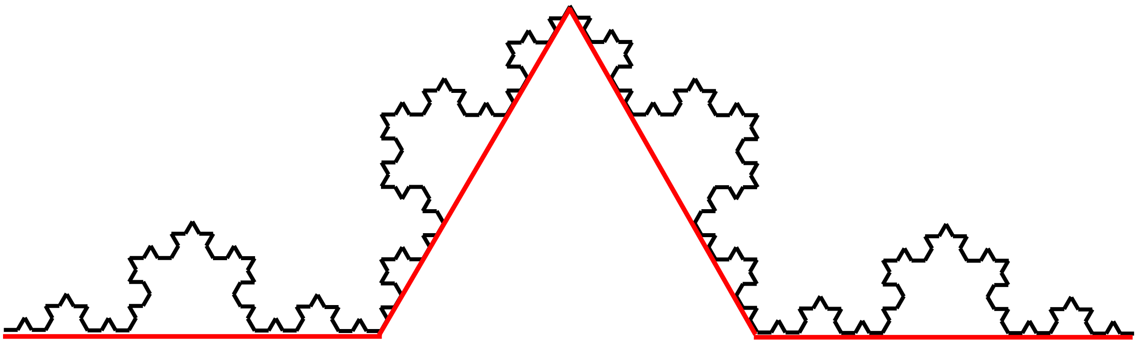
5- Measuring the Koch curve

Materials: Make your own ruler by cutting a strip of paper 15 cm = 1 unit long.
A compass would be useful too.



1 unit

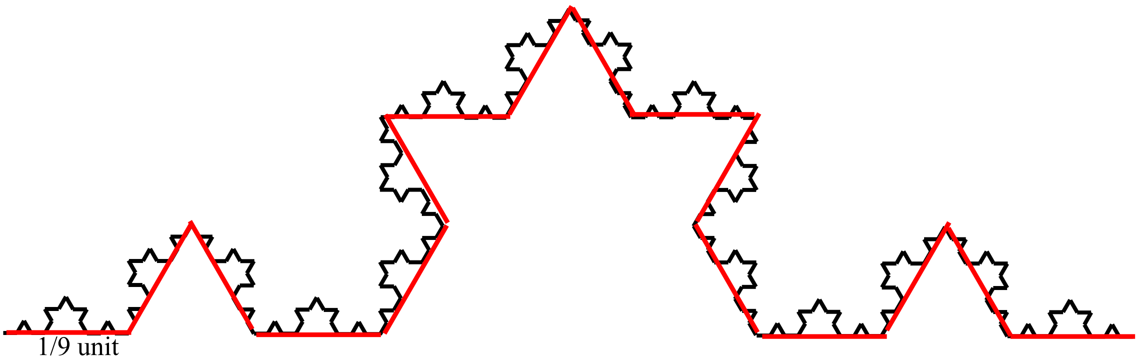
1. Folding your previous ruler into three parts measure the Koch curve.



1/3 unit

How many 1/3 unit long rulers do you need? **4 rulers**

2. How many 1/9 unit long rulers do you need to measure the Koch curve? **16 rulers**



1/9 unit

3. Do you see a pattern here?

If we use a ruler whose length is $1/3^2$, then we must use exactly 4^2 of these rulers.

4. Complete the grid below:

	Ruler length	**Num of rulers**
Stage 0	1	1
Stage 1	$\frac{1}{3} = 0.\overset{\frown}{3}$	**4**
Stage 2	$\left(\frac{1}{3}\right)^2 = \frac{1}{9} = 0.\overset{\frown}{1}$	$\mathbf{4^2 = 16}$
Stage 3	$\left(\frac{1}{3}\right)^3 = \frac{1}{27} =$ **0.0370…**	$\mathbf{4^3 = 64}$
Stage 4	$\left(\frac{1}{3}\right)^4 = \frac{1}{81} =$ **0.0123456790…**	$\mathbf{4^4 = 256}$

The smaller the ruler, the more rulers we need to measure the length of the Koch curve.

Notice that in the next questions we linearise data avoiding logarithms. Pupils can experiment with the log key in their calculators without knowing this specific concept.

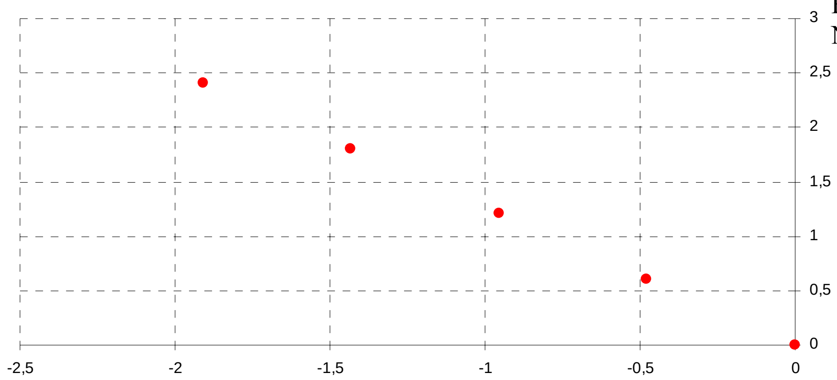
5. Experiment with the **exponent key 10^x** of your calculator to **rewrite in base 10** the entries in the table above (**correct** to two decimal places):

	Ruler length	**Num of rulers**
Stage 0	10^0	10^0
Stage 1	$10^{-0.47}$	$\mathbf{10^{0.60}}$
Stage 2	$\mathbf{10^{-0.95}}$	$\mathbf{10^{1.20}}$
Stage 3	$\mathbf{10^{-1.43}}$	$\mathbf{10^{1.80}}$
Stage 4	$\mathbf{10^{-1.90}}$	$\mathbf{10^{2.40}}$

6. Complete the table:

Exponent of ruler length in base 10	0	− 0.47	**− 0.95**	**− 1.43**	**− 1.90**
Exponent of Num of rulers in base 10	0	**0.60**	**1.20**	**1.80**	**2.40**

7. Sketch the graph **of the exponents** in the table above:



Exponent of ruler length in base 10

8. - Do these points lie on a straight line? **It would appear so.**

- Calculate the ratio $\frac{Exponent\ of\ \ Num.\ of\ \ rulers\ \ in\ \ base\ 10}{Exponent\ \ of\ \ ruler\ \ lenght\ \ in\ \ base\ 10}$

$\frac{0.60}{-0,47} \approx -\mathbf{1.27}$ $\frac{1.20}{-0.95} \approx -\mathbf{1.26}$ $\frac{1.80}{-1.43} \approx -\mathbf{1.25}$ $\frac{2.40}{-1.90} \approx -\mathbf{1.26}$

Actual plotted points do not all lie on a straight line, but they are very close to one.

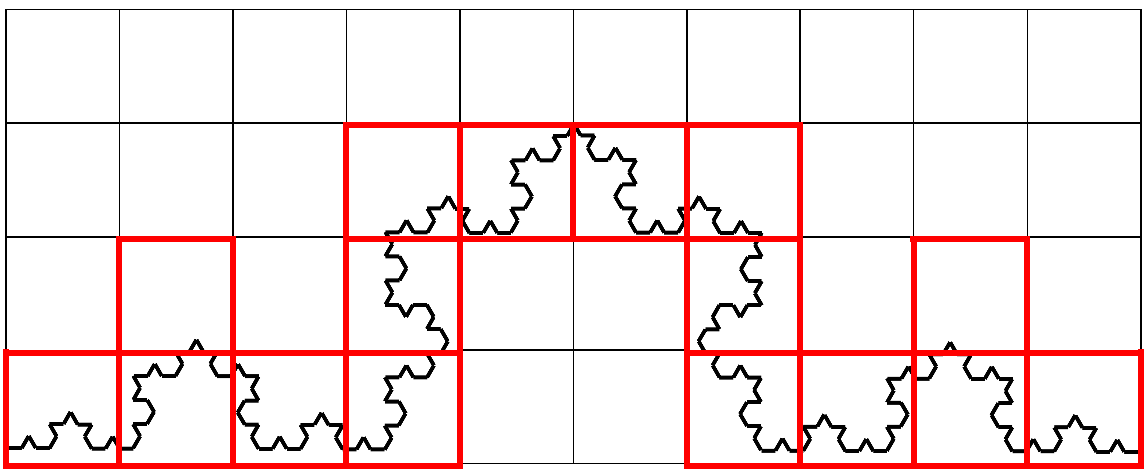
You can take this opportunity to talk about direct proportion and slope of a straight line.

- Do you recognise these values?

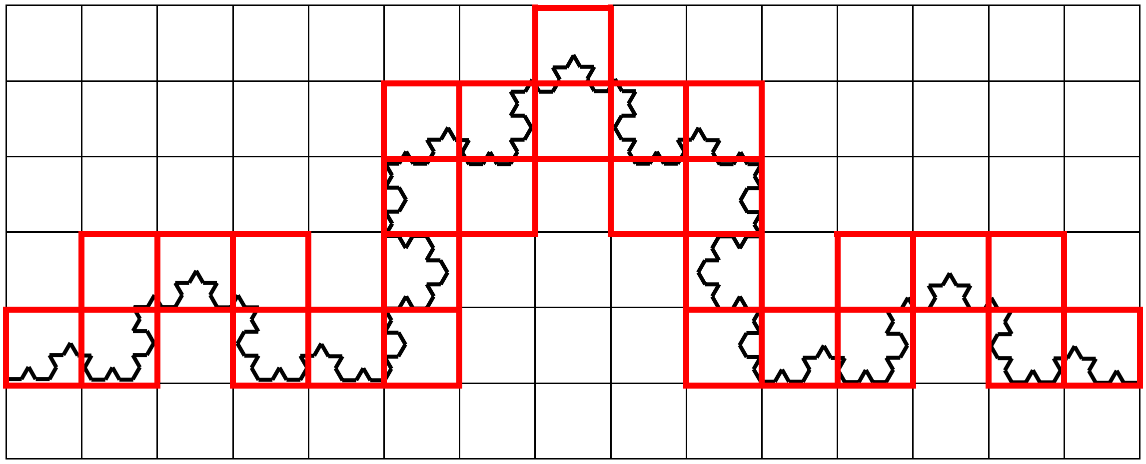
Sign changing we get an approximation of the fractal dimension of the Koch curve!

6- The Box Counting method

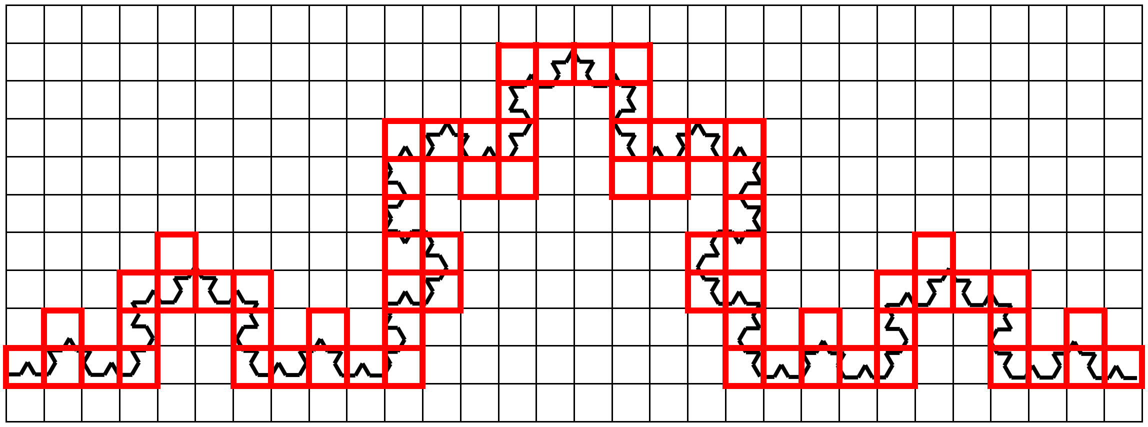
1. Paint the boxes that contain some of the Koch curve:



10-sized grid



15-sized grid



30-sized grid

2. Count the number of painted boxes and fill in the table below:

Grid size	10	15	30
Num. of boxes needed to cover the Koch curve	16	26	68

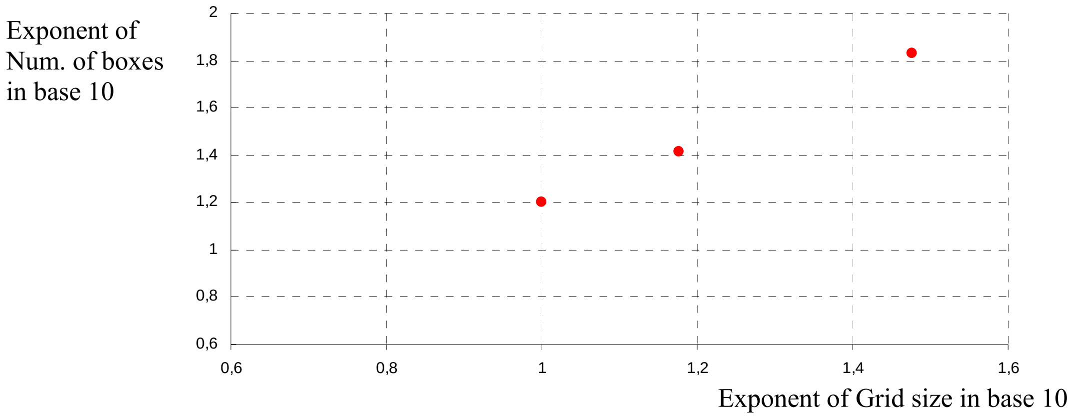
3. Experiment with the **exponent key 10** ^{x} of your calculator to **rewrite in base 10** the entries in the table above (**correct** to two decimal places):

Grid size	10^{1}	$10^{1.17}$	$10^{1.47}$
Num. of boxes needed to cover the Koch curve	$10^{1.20}$	$10^{1.41}$	$10^{1.83}$

4. Complete the table:

Exponent of Grid size in base 10	1	**1.17**	**1.47**
Exponent of Num. of boxes in base 10	**1.20**	**1.41**	**1.83**

5. Sketch the graph **of the exponents** in the table above:



6. - Do these points lie on a straight line? **It would appear so.**

- Calculate the ratio $\dfrac{Exponent\ \ of\ \ Num.\ of\ \ boxes\ \ in\ \ base\ 10}{Exponent\ \ of\ \ Grid\ \ size\ \ in\ \ base\ 10}$

$\dfrac{1.20}{1} \approx$ **1.20** $\dfrac{1.41}{1.17} \approx$ **1.20** $\dfrac{1.83}{1.47} \approx$ **1.24**

- Do you recognise these values?

We get an approximation of the fractal dimension of the Koch curve!