LESSON PLAN 6. Natural Fractals

Aim: To explore hands on fractal dimension of some natural objects. Fractals arise in many more areas than just mathematics and can be used as mathematical models of many of the complicated shapes that appear in nature.

Teaching objectives:

Content
- Geometric iteration rule
- Geometric transformations. Rotation.
- Copies of copies
- Self-similarity
- Fractal dimension
- The Ruler method
- The Box Counting method

Communication
- Predicting: “will be...”, “would be ...”,
- Comparing: “magnified by a factor of ...“
- Describing your work: “I have chosen ...”,...
- Arguing with partners predictions and conclusions
- Translating visual information into verbal descriptions
- Checking calculations in cooperative groups
- Sharing opinions to organise a poster
- Reading summary sentences to other groups
- Writing a script for an oral presentation

Cognition
- Identifying iteration rules
- Identifying similar shapes
- Developing intuition for fractal dimension
- Estimating dimensions through iterative procedures
- Predicting eventual outcomes
- Creating figures through drawing programs
- Linearising data
- Plotting a graph to illustrate selected information
- Writing summary reports using key vocabulary
- Arguing and checking answers

Context
- Images development
- Zooms of the size of images
- Computer graphics and visual art
- Modelling natural fractals

Outcomes: At the end of the lesson, pupils will be able to

- Estimate dimensions through iterative procedures
- Identify numerical and geometric patterns in different contexts
- Translate visual information to verbal descriptions
- Use drawing programs to generate further iterative outputs
- Predict what will happen if we continue iterating
- Be confident in geometric English language
- Develop a sense of responsibility when working in a cooperative group
- Recognise the importance of arguing in a team
- Know how to behave in a discussion
- Know how to make a script for an oral presentation
- Present their work to the rest of the class

Key words: Self-similarity, fractal dimension, Ruler method, Box Counting method.

Prerequisites: Pupils need to be familiar with self-similarity, fractal dimension, Ruler method, Box Counting method from the previous units.

Curriculum connections:

- Positive and negative powers of 10.
- Computation with exponents in a geometric context.
- Dimension.
- Practise with recognizing and describing numerical and geometric patterns.
- Geometric transformations. Similarity.
- Cartesian coordinates and graph plotting.
- Linear functions.
- Use of calculator or computer when needed.

Timing: Six 55-minute class periods.

Tasks planned:

First session

- Pupils work through 1-*Plant-growth.*
- Plenary: comments on self-similarity and fractal dimension.
- Homework: explore another plant growing.

Second session

- Homework revision: Each pupil presents to the rest of the class his/her home project.
- Ask pupils to start working with exercise 2-*Leaf outline* and to finish it as homework.

Third session

- Homework revision: Sharing results and final conclusions about the fractal dimension estimated.
  You can take this opportunity to revisit direct proportion and slope of a straight line.
- Cooperative groups start making a poster following the instructions from exercise 3-*Further exploration*.

Fourth session

- Plenary: poster presentation and oral process explanation.
- Pupils work through exercise 4- *Cauliflower* in cooperative groups. This exercise reinforces self-similarity.
- Homework: Finish the remaining calculations.

Fifth session

- Homework revision: Sharing results and final conclusions about the fractal dimension estimated.
- Provide pupils with worksheet 5-*Coastline*. Cooperative groups work accurately trough this exercise. Compasses would be useful.
- Homework: Ask pupils complete the remaining questions.

Sixth session

- Homework revision: Sharing results and final conclusions about the fractal dimension estimated. You can take this opportunity to talk about line of the best fit.
- Pupils work through exercise 6- *Bone cross-section* to test their feeling for fractal dimension.
- Plenary: Summary discussion.
- Invite your pupils to search for natural fractals in the Internet.

Resources

- For class activities and homework: the pupils' worksheets and access to a computer drawing program.
- For practising *The Ruler and the Box Counting method*s: a strip of paper, ruler and compasses, transparent square grids, pencil and rubber, coloured pencils, calculator…
- For making posters: leaves of different shapes from the park next to your school, cardboard, scissors and glue, pen, markers ...
- For estimating self-similarity and fractal dimension: cauliflower or broccoli, knife and magnifying glass.
- For teaching instructions and answer keys to each exercise: the teachers' worksheets, mainly borrowed from the book *Fractals* by J. Choate, R.L. Devaney, A. Foster, listed in the bibliography.

Assessment

- The teacher will take notes of pupils' progress every period.
- Good presentation in worksheets and active participation in the classroom is essential.
- Posters will be individually marked depending on participation, contribution to the group work and oral presentation.
- Homework will be suitably marked.
- Pupils who talk in English will be given extra marks.

Evaluation

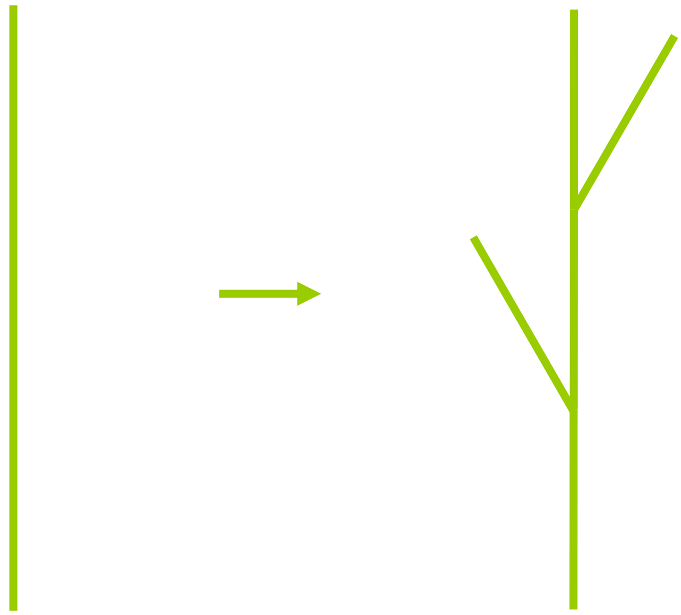
TEACHER'S NOTES AND ANSWERS

Answers to exercises

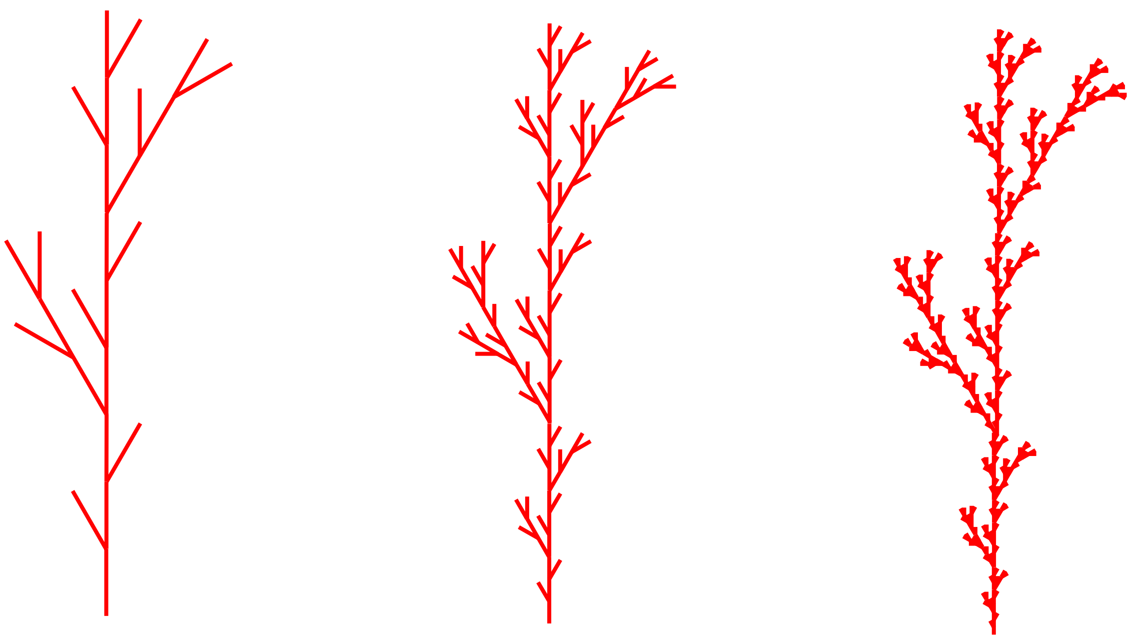
1-Plant-growth

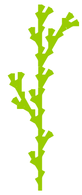
1. Use any computer **drawing program** in this exercise.
Start with a straight line segment. **Everyone** in class should start with the **same long** segment.
Apply the following iteration rule:

- Reduce the segment by one third and make five copies.
- Rotate one copy 30º clockwise and another 30º anticlockwise.
- Assemble in this way:



Now draw the second, the third and the fourth iteration.





2. If we continue iterating we get a fractal plant.
- To investigate self-similarity and its fractal dimension, fill in the blanks:

After magnifying by a factor of 3 we get **five** copies of the original plant.

So the fractal dimension will be the exponent *d* such that $3^d = 5$.

- Experiment with the **exponent key** of your calculator to find out the fractal dimension (**correct** to two decimal places).

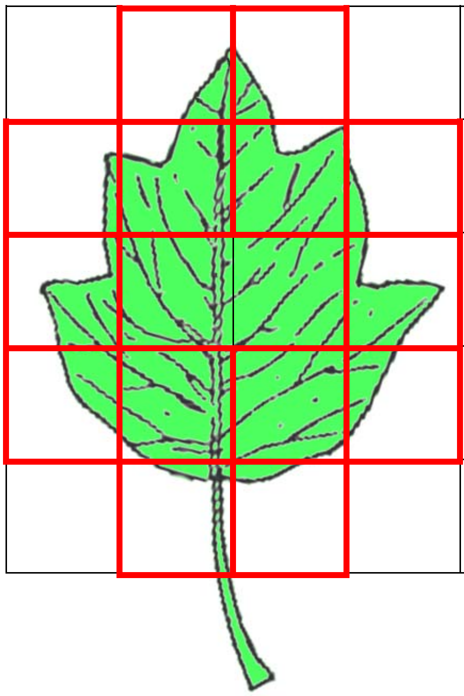
As $3^1 = 3$ and $3^2 = 9$, the dimension *d* will be a number between 1 and 2.
Let's try
$3^{1.3} \approx 4.171167510$ $3^{1.4} \approx 4.655536721$ $3^{1.5} \approx 5.196152422$ $3^{1.47} \approx 5.027687135$
$3^{1.46} \approx 4.972754646$
The fractal dimension will be approximately 1.46, correct to two decimal places.

- Compare to the fractal dimension of the Koch curve.

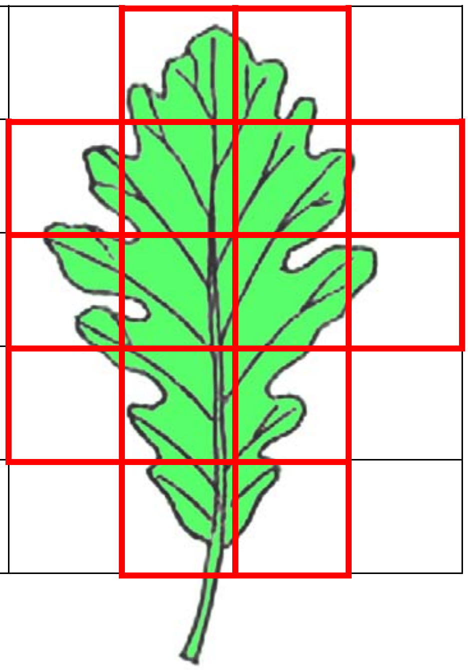
The fractal dimension of the Koch curve is smaller than the fractal dimension of this plant. This is just as we could expect, as the Koch curve appears to be less thick than the plant.

2-Leaf outline

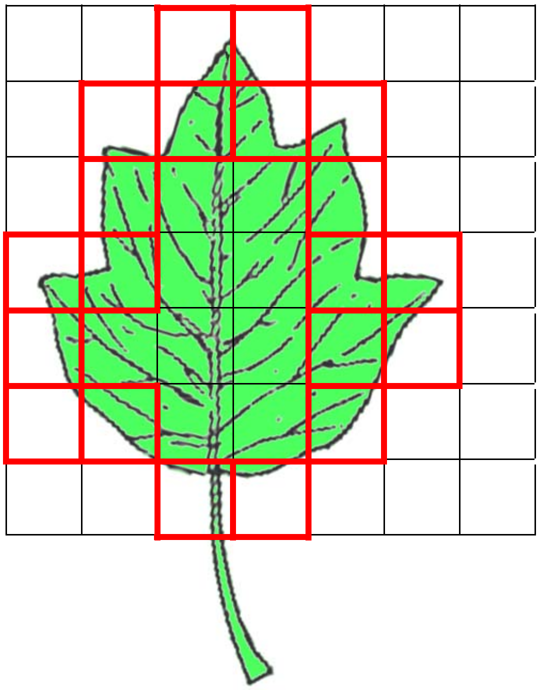
1. Paint the boxes below that contain some of the leaf outlines:



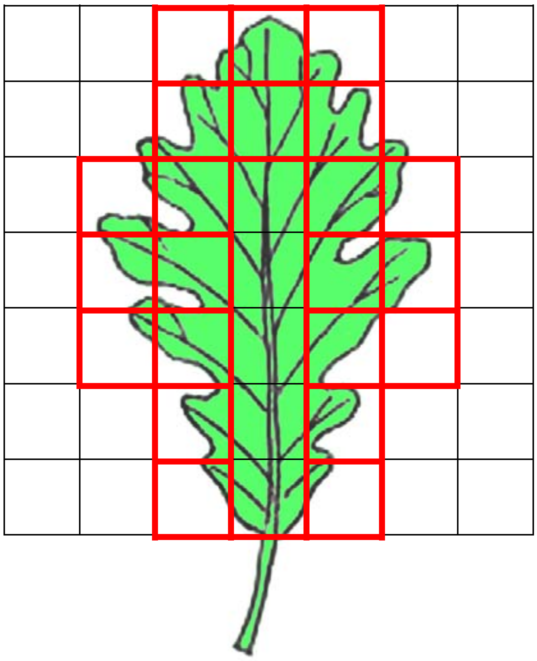
5-sized grid



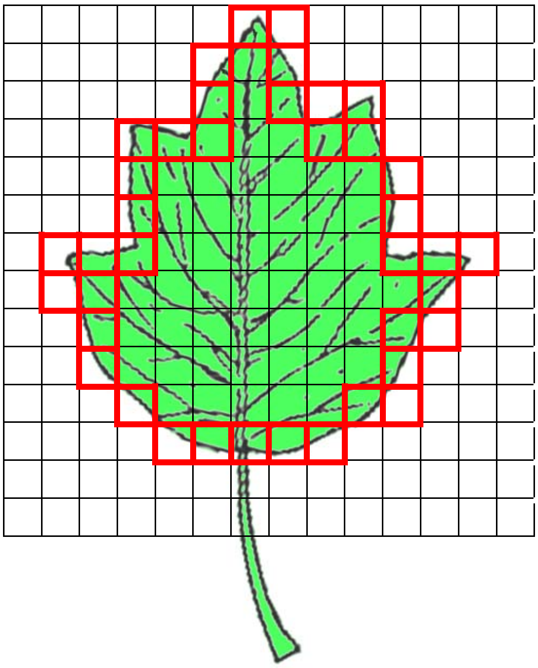
5-sized grid



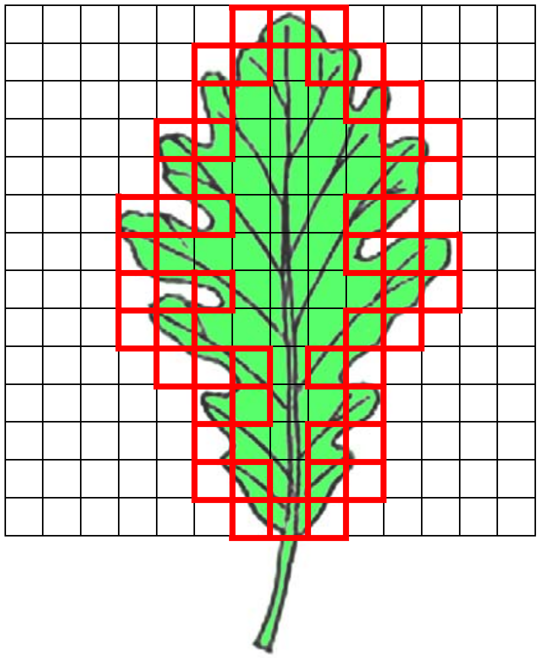
7-sized grid



7-sized grid



14-sized grid



14-sized grid

2. Count the number of painted boxes and fill in the table below:

Grid size	5	**7**	**14**
Num. of boxes covering the leaf contour 1	**14**	**20**	**39**
Num. of boxes covering the leaf contour 2	**15**	**22**	**53**

3. Experiment with the **exponent key 10** ^{x} of your calculator to **rewrite in base 10** the entries in the table above (**correct** to two decimal places):

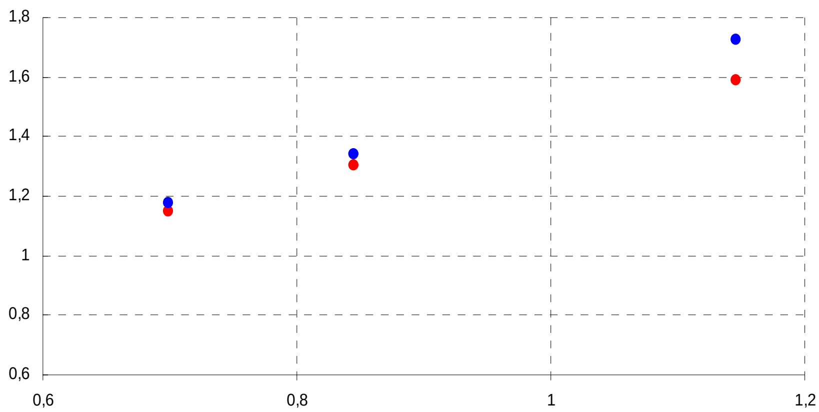
Grid size	$10^{0.69}$	$10^{0.84}$	$10^{1.14}$
Num. of boxes covering the leaf contour 1	$10^{1.14}$	$10^{1.30}$	$10^{1.59}$
Num. of boxes covering the leaf contour 2	$10^{1.17}$	$10^{1.34}$	$10^{1.72}$

4. Complete the table:

Exponent of Grid size in base 10	0.69	**0.84**	**1.14**
Leaf 1:Exponent of Num. of boxes in base 10	**1.14**	**1.30**	**1.59**
Leaf 2:Exponent of Num. of boxes in base 10	**1.17**	**1.34**	**1.72**

5. Sketch the graphs **of the exponents** in the table above (use two colours):

Exponent of Num. of boxes in base 10



Exponent of Grid size in base 10

Legend: Leaf1 Leaf 2

6. Calculate the ratio $\dfrac{Exponent\ \ of\ \ Num.\ of\ \ boxes\ \ in\ \ base\ \ 10}{Exponent\ \ of\ \ Grid\ \ size\ \ in\ \ base\ \ 10}$

Leaf 1:

$\dfrac{1.14}{0.69} \approx \mathbf{1.65}$ $\dfrac{1.30}{0.84} \approx \mathbf{1.54}$ $\dfrac{1.59}{1.14} \approx \mathbf{1.39}$

Leaf 2:

$\dfrac{1.17}{0.69} \approx \mathbf{1.69}$ $\dfrac{1.34}{0.84} \approx \mathbf{1.59}$ $\dfrac{1.72}{1.14} \approx \mathbf{1.50}$

You can take this opportunity to talk about direct proportion and slope of a straight line.

7. What are the fractal dimensions of these leaf outlines?

We would say 1.5 and 1.6 approximately, but only three measures are not enough to quantify the fractal dimension.

8. What does the fractal dimension measure? Argue with your partner.

Fractal dimension estimates the complexity of leaf outline. More complex leaf outlines have higher fractal dimension. As we see in our leaves, the contour of the second leaf is more complex than the contour of the first one. Fractal dimension quantifies this complexity.

3-Further exploration

Cooperative group: Three or four classmates.

Materials:
- Leaves of different shapes from the park next to your school.
- Different sized square grids printed on transparencies.
- Calculator, cardboard, glue, pen, coloured pencils, markers…

Work: Calculate the fractal dimension of the outline of your leaves.

Poster presentation:
- Order your leaves by fractal dimension from the lowest to the highest.
- Glue them on a cardboard.
- Write down its fractal dimension.
- Present your poster to the rest of the class.

Go to: http://hypertextbook.com/facts/2002/leaves.shtml for more information.

4-Cauliflower

Cauliflower  Broccoli 

- Cauliflower and broccoli are fractals because they branch off into smaller and smaller pieces, which are similar in shape to the original.
Cooperative group: Three or four classmates.

Materials:
- Cauliflower or broccoli.
- Knife and magnifying glass.

Aim: Explore self-similarity and determine the fractal dimension of cauliflower and broccoli.

Procedure:
- Count the number of branches growing off the main trunk.
- Cut off these branches.
- They are copies of the whole. Estimate on average the magnification factor.
- Repeat the previous steps on all of these branches. Take the mean.
- Repeat again. Eventually, you will need a magnifying glass to see the last branches you cut off.

- Complete the sentence: **Open answer**
Every branch carries around ……………..……..branches …………....times smaller.

The fractal dimension will be approximately the exponent *d* such that

$$(\textit{Mean of number of branches})^d = \textit{Mean of magnification factors}$$

- Experiment with the **exponent key** of your calculator to find out the fractal dimension (**correct** to two decimal places).
Open answer between two and three.
- Compare your dimension to the dimension from other groups.

5-Coastline

Cooperative group: Three or four classmates.

Aim: Calculate the fractal dimension of *Estany de Banyoles* (*Catalonia*).
Other coastlines are possible.
Materials:
- Map of the coastline
- Make your own ruler by cutting a strip of paper 16 cm = 1 unit long.

1 unit

- Compasses would be useful too.

Procedure: The Ruler method.

- Measure the length of the coastline using a collection of rulers whose lengths get shorter and shorter.
- Rewrite the data as a power of 10.
- **To minimize error, average the results from your group.**
- Plot the exponent data "length of ruler versus number of rulers needed".
- You get a set of points that is almost linear.

1 unit

1/2 units

1/4 units

1/8 units

1/16 units

1:10.000

1. After measuring the coastline from the map below, complete the table:

	Ruler length	Mean of Num of rulers
Stage 0	1	1
Stage 1	$\frac{1}{2} = 0.5$	**3.8**
Stage 2	$\left(\frac{1}{2}\right)^2 = \frac{1}{4} = 0.25$	**8.7**
Stage 3	$\left(\frac{1}{2}\right)^3 = \frac{1}{8} = \mathbf{0.125}$	**19.8**
Stage 4	$\left(\frac{1}{2}\right)^4 = \frac{1}{16} = \mathbf{0.0625}$	**40**

The smaller the ruler, the more rulers we need to measure the coastline.

Notice that in the next questions we linearise data avoiding logarithms. Pupils can experiment with the log key in their calculators without knowing this specific concept.

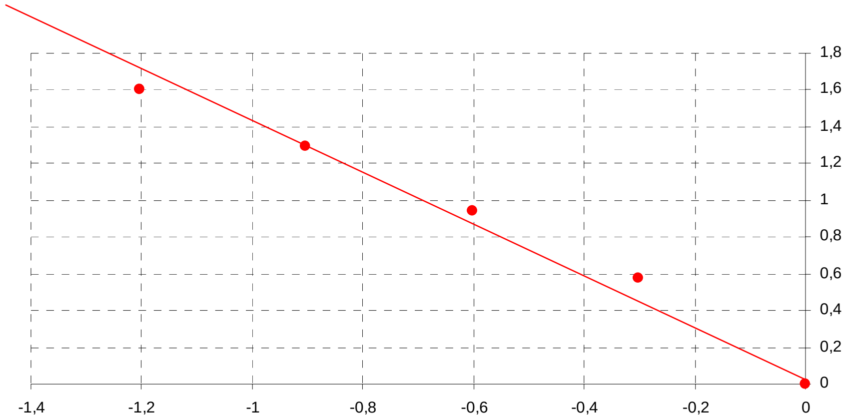
2. Experiment with the **exponent key 10** ^{x} of your calculator to **rewrite in base 10** the entries in the table above (**correct** to two decimal places):

	Ruler length	Num of rulers
Stage 0	10^0	10^0
Stage 1	$10^{-0.30}$	$\mathbf{10^{0.57}}$
Stage 2	$\mathbf{10^{-0.60}}$	$\mathbf{10^{0.93}}$
Stage 3	$\mathbf{10^{-0.90}}$	$\mathbf{10^{1.29}}$
Stage 4	$\mathbf{10^{-1.20}}$	$\mathbf{10^{1.60}}$

3. Complete the table:

Exponent of ruler length in base 10	0	− 0.30	**− 0.60**	**− 0.90**	**− 1.20**
Exponent of Num of rulers in base 10	0	**0.57**	**0.93**	**1.29**	**1.60**

4. Sketch the graph **of the exponents** in the table above:



Exponent of Num. of rulers in base 10

Exponent of ruler length in base 10

5. Calculate the ratio $\dfrac{Exponent\ of\ \ Num.\ of\ \ rulers\ in\ base\ 10}{Exponent\ of\ \ ruler\ \ lenght\ \ in\ base\ 10}$

$\dfrac{0.57}{-0{,}30} \approx \mathbf{-1.88}$ $\dfrac{0.93}{-0.60} \approx \mathbf{-1.56}$ $\dfrac{1.29}{-0.90} \approx \mathbf{-1.43}$ $\dfrac{1.60}{-1.20} \approx \mathbf{-1.33}$

Actual plotted points do not all lie on a straight line, but they are very close to one.

You can take this opportunity to talk about direct proportion, slope of a straight line and line of best fit..

6. What is the fractal dimension of *Estany de Banyoles* coastline?

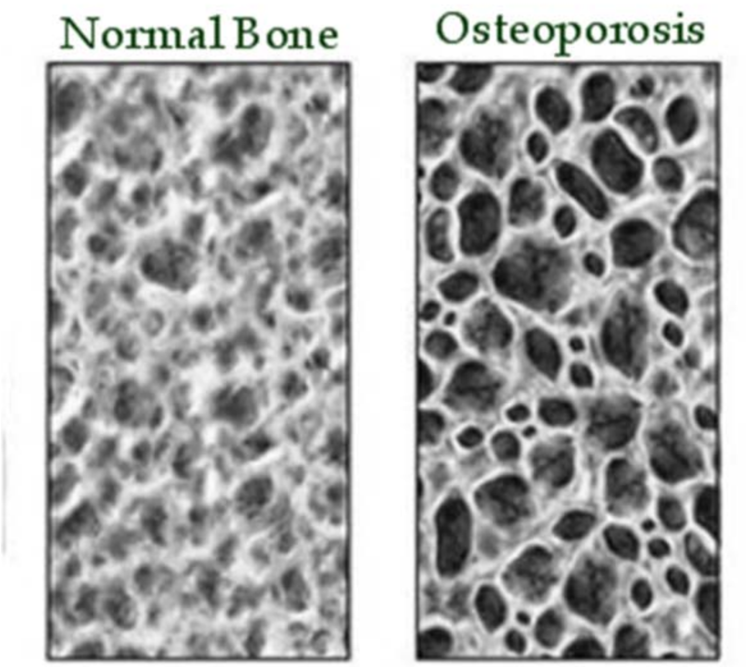
We do not get the exact value of the fractal dimension by this method. We only get an approximation. **Our solution is about 1.43.**

Pupils can get a better solution using a compass with a very sharp pencil, running two or three trials with the same ruler and averaging the results.

Next exercise is to test intuition for fractal dimension.

6-Bone cross-section

- Look at the digital images of a cross-section of bones



- Using the box counting method it is known that cross-section of normal bones have fractal dimension between 1.7 and 1.8.

- Osteoporosis is a condition of decreased bone mass. This leads to fragile bones which are at an increased risk for fractures.

- Choose the word Lower or Bigger to complete the sentence:
Lower fractal dimension is an index for the early diagnosis of osteoporosis.
Notice that the second image has many more holes than the first one.