HOW TO EXPLAIN AFFINE POINT GEOMETRY[1]

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Abstract

Hermann Grassmann based his extension theory on Möbius' barycentric calculus [1]. According to Grassmann, a line is the exterior product of two points, a plane is the exterior product of three points, and an extension having what we now call n dimension is the exterior product of n+1 points [2]. Also, the exterior product of a line and a point is a plane, and the exterior product of two non-intersecting lines is the affine space. How should we explain this Grassmann's point geometry to our pupils instead of the usual vector geometry? The algebraic way to teach it [3] is by using barycentric and homogeneous coordinates [4]. It will be shown that they have many advantages such as the natural introduction to projective geometry and duality, which become trivial when they are understood by means of pencils of lines and sheaves of planes.

The affine space

The affine space A_n with dimension n is a set $\{E_n, V_n, +\}$ where E_n is a point space and V_n a vector space of dimension n and $+$ is the affine mapping:

$$+: \; E \times V \to E$$
$$(P, v) \to Q = P + v$$

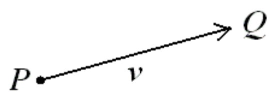

Figure 1

The affine mapping maps a point P and a vector v into another point Q obtained from P by means of the translation given by the vector v (figure 1). Then, the translation which maps the point P into the point Q is obtained by subtraction of their coordinates:

$$v = Q - P$$

Coordinate systems in the three-dimensional affine space

A coordinate system is a set $\{O; e_1, e_2, e_3\}$, where O is a point of the geometric space E_3 called the *origin of coordinates* and $\{e_1, e_2, e_3\}$ are three independent vectors of the three-dimensional space V_3. By using the affine mapping, any point P of the geometric space is written in a unique way as:

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[1] This talk was given at the 10th International Conference on Clifford Algebras and their Applications in Mathematical Physics (ICCA 10) held in Tartu (Estonia) from August 4th to 9th, 2014 during the workshop *Geometric Algebra in the High School and Undergraduate Curriculum.*

$$P = O + x\,e_1 + y\,e_2 + z\,e_3 \qquad \Leftrightarrow \qquad OP = x\,e_1 + y\,e_2 + z\,e_3 \qquad x, y, z \in \mathbb{R}$$

where OP is the position vector of P. The components (x, y, z) are called the *coordinates* of P in the coordinate system $\{O; e_1, e_2, e_3\}$[2].

Four non-coplanar points $\{O, A, B, C\}$ of the three-dimensional space (figure 2) always determine a coordinate system with O as the origin and basis vectors:

$$e_1 = OA \qquad e_2 = OB \qquad e_3 = OC$$

By substitution of the basis vectors in the equation of the coordinate system we have:

$$OP = x\,OA + y\,OB + z\,OC$$

and undoing vectors into points we have:

$$P = (1 - x - y - z)\,O + x\,A + y\,B + z\,C \qquad x, y, z \in \mathbf{R} \qquad O, A, B, C, P \in \mathbf{R}^3$$

where $(1 - x - y - z, x, y, z)$ are the *barycentric coordinates*. That is, any point in the geometric space is equal to a linear combination of the four basis points whose coefficients' addition is the unity. On the other hand, any vector can be written as a linear combination of four non coplanar points whose coefficients' addition is the zero:

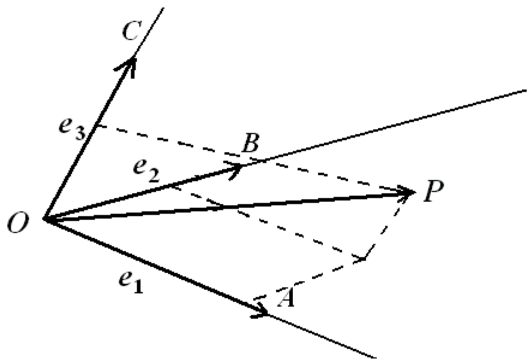


Figure 2

$$v = (-x - y - z)\,O + x\,A + y\,B + z\,C$$

$$= x\,OA + y\,OB + z\,OC = x\,e_1 + y\,e_2 + z\,e_3$$

Points at infinity

A point R on the line $\overline{PQ}$ can be written as $R = (1-k)P + k\,Q$, where k is its barycentric coordinate on the line $\overline{PQ}$. If $k = 0$ then $R = P$. If $k = 1$ then $R = Q$. If $0 < k < 1$ then R runs over the points on the segment PQ (figure 3).

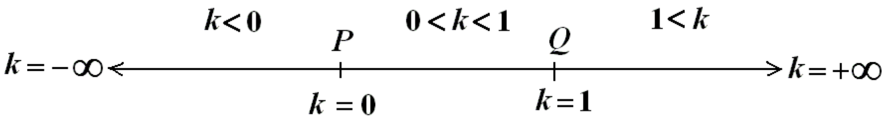


Figure 3

[2] Here x, y, z are coordinates given in a generic coordinate system that is not necessarily orthonormal. Since we set aside the notion of perpendicularity, all the equations here outlined have general validity for any coordinate system.

If $k=\infty$ then R is a point at infinite distance from P and Q, that is, a point at infinity. In this case, the unity can be disregarded in comparison with infinity so that we can write:

$$R=-kP+kQ \qquad\qquad k\rightarrow\infty$$

Then, the characteristic of the points at infinity is the fact that the addition of their coordinates is zero.

Homogeneous coordinates

The homogeneous coordinates allow us to incorporate the points at infinity. Points at infinity are distinguished from points at a finite distance from the barycentre of the point basis because the addition of their point coordinates is zero. Therefore, their coordinates cannot be normalized. It is then assumed that the coordinates can be multiplied by any factor k and they continue to represent the same point:

$$(1-x-y-z, x, y, z)=(k(1-x-y-z), kx, ky, kz) \qquad\qquad k\in\mathbb{R}-\{0\}$$

For instance $(2, 3, -1, 1)=(0.4, 0.6, -0.2, 0.2)$. The addition of the latter coordinates is the unity so that they are barycentric and they correspond to the Cartesian coordinates $(x, y, z)=(0.6, -0.2, 0.2)$. On the other hand the point $Q(4, -6, 1, 1)$ is a point at infinity because the addition of their homogeneous coordinates is null. It is the point located in the direction of the vector:

$$Q=4O-6A+B+C=-6OA+OB+OC=-6e_1+e_2+e_3$$

If we multiply the coordinates of P by any constant, the corresponding vector is multiplied by the same constant, but the point at infinity does not change because vectors that are proportional indicate the same direction, that is, the same point at infinity.

Basis of extensions with higher grades

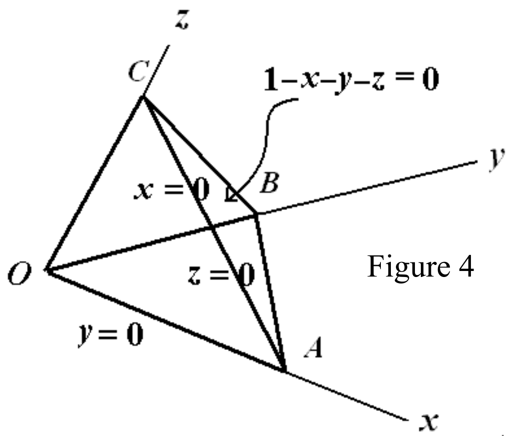


Figure 4

The fundamental tetrahedron $OABC$ also defines a basis of lines and planes. The lines $\{\overline{OA}, \overline{OB}, \overline{OC}, \overline{AB}, \overline{BC}, \overline{AC}\}$ containing the edges of the tetrahedron are a basis for all the lines of the space (figure 4). In the same way, the planes containing the tetrahedron faces $\{OAB, OBC, OAC, ABC\}$ are a basis for the planes in the space as I explain below.

The plane $\overline{OAB}$ has the equation $z=0$; the plane $\overline{OBC}$ has the equation $x=0$; the plane $\overline{OAC}$ has the equation $-y=0$

and the plane $\overline{ABC}$ has the equation $-(1-x-y-z=0)$. We have taken the same orientation of the planes in order to give consistence.

The exterior product of points

According to Hermann Grassmann, the exterior product of two points yields a line. Let us think of a line passing through points P and Q. We can arrange their barycentric coordinates in a matrix:

$$\begin{pmatrix} t_P & x_P & y_P & z_P \\ t_Q & x_Q & y_Q & z_Q \end{pmatrix} \qquad \text{where } t = 1 - x - y - z$$

The exterior product of both points is then:

$$PQ = \begin{vmatrix} t_P & x_P \\ t_Q & x_Q \end{vmatrix} OA + \begin{vmatrix} t_P & y_P \\ t_Q & y_Q \end{vmatrix} OB + \begin{vmatrix} t_P & z_P \\ t_Q & z_Q \end{vmatrix} OC + \begin{vmatrix} x_P & y_P \\ x_Q & y_Q \end{vmatrix} AB + \begin{vmatrix} x_P & z_P \\ x_Q & z_Q \end{vmatrix} AC + \begin{vmatrix} y_P & z_P \\ y_Q & z_Q \end{vmatrix} BC$$

Of course, P or Q can be exchanged for some linear combinations of them and the new matrix still represents the same line:

$$\begin{pmatrix} t_P & x_P & y_P & z_P \\ (1-\lambda)t_P + \lambda t_Q & (1-\lambda)x_P + \lambda x_Q & (1-\lambda)y_P + \lambda y_Q & (1-\lambda)z_P + \lambda z_Q \end{pmatrix}$$

Observe that all the determinants are proportional under this change:

$$\begin{vmatrix} t_P & x_P \\ (1-\lambda)t_P + \lambda t_Q & (1-\lambda)x_P + \lambda x_Q \end{vmatrix} = \lambda \begin{vmatrix} t_P & x_P \\ t_Q & x_Q \end{vmatrix}$$

That is, we obtain now an exterior product with proportional components. Of course, the sum of the components of the original exterior product PQ is not the unity and they are therefore homogeneous coordinates. In order to obtain barycentric coordinates for lines, we must divide the calculated components by their addition. In conclusion, any points on a given line may be used and their exterior product gives the decomposition of the lines in the line basis of the space in a unique way.

Let us see an example. Let *r* be the line passing through the points *P*(2, 0, −3) and *Q*(3, −1, 4). Let us write the matrix of the barycentric coordinates of the line:

$$\begin{array}{cc} & \begin{array}{cccc} O & A & B & C \end{array} \\ r = & \begin{pmatrix} 2 & 2 & 0 & -3 \\ -5 & 3 & -1 & 4 \end{pmatrix} \end{array}$$

Then we have the minors:

$$\overline{OA} \to \begin{vmatrix} 2 & 2 \\ -5 & 3 \end{vmatrix} = 16 \qquad \overline{OB} \to \begin{vmatrix} 2 & 0 \\ -5 & -1 \end{vmatrix} = -2 \qquad \overline{OC} \to \begin{vmatrix} 2 & -3 \\ -5 & 4 \end{vmatrix} = -7$$

$$\overline{AB} \to \begin{vmatrix} 2 & 0 \\ 3 & -1 \end{vmatrix} = -2 \qquad \overline{AC} \to \begin{vmatrix} 2 & -3 \\ 3 & 4 \end{vmatrix} = 17 \qquad \overline{BC} \to \begin{vmatrix} 0 & -3 \\ -1 & 4 \end{vmatrix} = -3$$

The exterior product of P by Q has as components the minors of the matrix:

$$r = \overline{PQ} = 16\overline{OA} - 2\overline{OB} - 7\overline{OC} - 2\overline{AB} + 17\overline{AC} - 3\overline{BC}$$

The coefficients of this decomposition are homogeneous coordinates of r for the line basis $\{\overline{OA}, \overline{OB}, \overline{OC}, \overline{AB}, \overline{AC}, \overline{BC}\}$. We must divide by the sum of the coefficients in order to obtain the barycentric coordinates of r for this line basis:

$$r = \overline{PQ} = \frac{16}{19}\overline{OA} - \frac{2}{19}\overline{OB} - \frac{7}{19}\overline{OC} - \frac{2}{19}\overline{AB} + \frac{17}{19}\overline{AC} - \frac{3}{19}\overline{BC}$$

Therefore its barycentric coordinates are:

$$\overline{PQ} = \left(\frac{16}{19}, -\frac{2}{19}, -\frac{7}{19}, -\frac{2}{19}, \frac{17}{19}, -\frac{3}{19}\right)$$

Sheaves of planes

Let us consider a line r given by the intersection of two planes π_1 and π_2:

$$r : \begin{cases} \pi_1 : \quad p\,x + q\,y + r\,z + s = 0 \\ \pi_2 : \quad p'\,x + q'\,y + r'\,z + s' = 0 \end{cases}$$

A linear combination of both equations yields another plane containing the line:

$$(1-\lambda)(p\,x + q\,y + r\,z + s) + \lambda\,(p'\,x + q'\,y + r'\,z + s') = 0$$

The set of all the planes having this equation are the *sheaf of planes* of this line (figure 5). By repeating the linear combinations with other planes, we can interpret a plane as a linear combination of four non coplanar planes. For instance, let us consider the plane $\pi : 2x + 3y - z + 6 = 0$, whose equation is a linear combination of the equations of the planes π_1 and π_2:

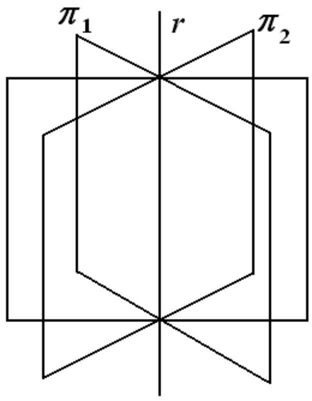


Figure 5

$(1-\lambda)\,\pi_1 + \lambda\,\pi_2$

$$\begin{cases} \pi_1: & 2x + 3y = 0 \\ \pi_2: & -z + 6 = 0 \end{cases}$$

$\pi = \pi_1 + \pi_2$ or $\pi = \dfrac{\pi_1 + \pi_2}{2}$ in barycentric coordinates

Therefore, π belongs to the sheaf of planes of the line intersection of π_1 and π_2. On the other hand, π_1 is a linear combination of the planes $x = 0$ and $y = 0$ so that π_1 belongs to the sheaf of planes of the z-axis $\overline{OC}$ (figure 3).

Barycentric equation of a plane

The Cartesian equation of a plane is:

$$p\,x + q\,y + r\,z + s = 0 \qquad p, q, r, s \in \mathbb{R}$$

By using barycentric coordinates we have:

$$s\,(1 - x - y - z) + (p + s)\,x + (q + s)\,y + (r + s)\,z = 0$$

which is a linear combination of the equations of the planes of the four faces of the fundamental tetrahedron:

$$1 - x - y - z = 0 \qquad x = 0 \qquad y = 0 \qquad z = 0$$

A proportional equation represents the same plane, so that the coefficients of this linear combination are homogeneous coordinates. Dividing by the sum of coordinates we have:

$$(1 - a - b - c)(1 - x - y - z) + a\,x + b\,y + c\,z = 0$$

with $a = \dfrac{p + s}{p + q + r + 4s}$ $\quad b = \dfrac{q + s}{p + q + r + 4s}$ $\quad c = \dfrac{r + s}{p + q + r + 4s}$

We can write $o = 1 - a - b - c$. Then:

$$o\,(1 - x - y - z) + a\,x + b\,y + c\,z = 0 \qquad o + a + b + c = 1$$

Then $[o, a, b, c]$ are the barycentric coordinates of the plane in the vector space of the planes with the plane basis $\{1 - x - y - z = 0, x = 0, y = 0, z = 0\}$. They are the *dual coordinates*. The plane is a point in the vector space of the planes. Its Cartesian coordinates are $[a, b, c]$.

Let us see an example. Let us take the plane:

$$2x-5y+4z+3=0 \quad \Leftrightarrow \qquad 3(1-x-y-z)+5x-2y+7z=0$$

The sum of the homogeneous dual coordinates are $3+5-2+7=13$. Therefore, the plane equation can be written as:

$$\frac{3}{13}(1-x-y-z)+\frac{5}{13}x-\frac{2}{13}y+\frac{7}{13}z=0$$

and the barycentric dual coordinates are $[3/13, 5/13, -2/13, 7/13]$. Its Cartesian dual coordinates are $[5/13, -2/13, 7/13]$.

Matrix equation of a line

A line is the intersection of two planes. In barycentric coordinates:

$$\begin{cases} o\,(1-x-y-z)+a\,x+b\,y+c\,z=0 \\ o'\,(1-x-y-z)+a'\,x+b'\,y+c'\,z=0 \end{cases}$$

We can write this equation system in matrix form:

$$\begin{pmatrix} 1-x-y-z & x & y & z \end{pmatrix}\begin{bmatrix} o & o' \\ a & a' \\ b & b' \\ c & c' \end{bmatrix}=0$$

Both are matrices, but parentheses indicate point coordinates and square brackets indicate dual coordinates. Any point on the line will fulfil this equation, so that we can add another point:

$$\begin{pmatrix} 1-x-y-z & x & y & z \\ 1-x'-y'-z' & x' & y' & z' \end{pmatrix}\begin{bmatrix} o & o' \\ a & a' \\ b & b' \\ c & c' \end{bmatrix}=0$$

Putting $t=1-x-y-z$ we have:

$$\begin{pmatrix} t & x & y & z \\ t' & x' & y' & z' \end{pmatrix}\begin{bmatrix} o & o' \\ a & a' \\ b & b' \\ c & c' \end{bmatrix}=0$$

This is the *matrix equation* of a line. On the other hand, note that there is no need for the addition of coordinates to be the unity so that we can also use homogeneous coordinates. If we change any row of the first matrix by a linear combination of both rows, the equation is preserved. This algebraic operation consists of the substitution of the first point by another point on the line. If we change any column of the second matrix by a linear combination of both columns, the equation is also preserved. This algebraic operation consists of the substitution of the first plane by another plane of the sheave of planes of the line.

Decomposition of a line given two planes containing it

We can find out the components of a line for the line basis from the planes determining it instead of taking two points on the line. I have proven that this decomposition yields exactly the same result provided that the suitable correspondence between planes and their equations is taken.

Let us take the line of the former example of decomposition, which passes through the points $P(2, 0, -3)$ and $Q(3, -1, 4)$. Its direction vector is $v = Q - P = (1, -1, 7)$ so that the continuous equation of the line is:

$$\frac{x-2}{1} = \frac{y}{-1} = \frac{z+3}{7}$$

Separating both equalities we obtain two planes:

$$r:\begin{cases} \pi_1: & x + y - 2 = 0 \\ \pi_2: & 7y + z + 3 = 0 \end{cases}$$

Let us pass to barycentric coordinates:

$$r:\begin{cases} \pi_1: & -2(1 - x - y - z) - x - y - 2z = 0 \\ \pi_2: & 3(1 - x - y - z) + 3x + 10y + 4z = 0 \end{cases}$$

Therefore the matrix form of r in dual coordinates is:

$$r = \begin{bmatrix} -2 & 3 \\ -1 & 3 \\ -1 & 10 \\ -2 & 4 \end{bmatrix} \begin{matrix} 1 - x - y - z = 0 \\ x = 0 \\ y = 0 \\ z = 0 \end{matrix}$$

The equation of each plane is indicated on the right. If we make the exterior product of the dual coordinates of both planes we will obtain the decomposition of the line in the line basis. Each line of the basis is the intersection of two of the fundamental planes. In order to determine the sign of the fundamental line that corresponds to a pair of plains, we can apply the screw rule: when turning the vector perpendicular to the first plane towards the vector perpendicular to the second plane as it were the head of a screw, the sense of advancement of this screw will be the sense of the resulting line. For example, the intersection of the planes $x = 0$ and $y = 0$ gives as a result the line $\overline{OC}$ because when turning the screw top from OA up to OB the screw will move in the sense of OC

(figure 6). The intersection of the planes $1-x-y-z=0$ and $x=0$ gives as a result the line $\overline{BC}$ because when turning the screw top from the vector $-OA-OB-OC$, perpendicular to the first plane, up to the vector OA the screw will move in the sense of BC (figure 7):

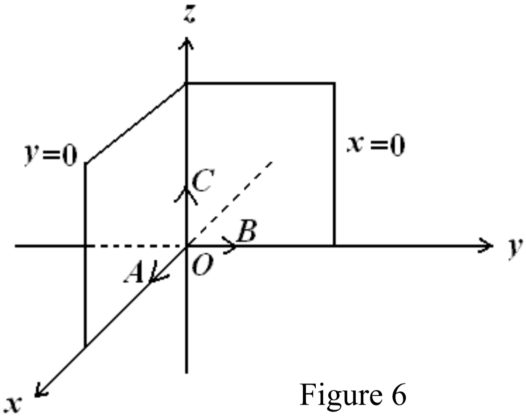


Figure 6

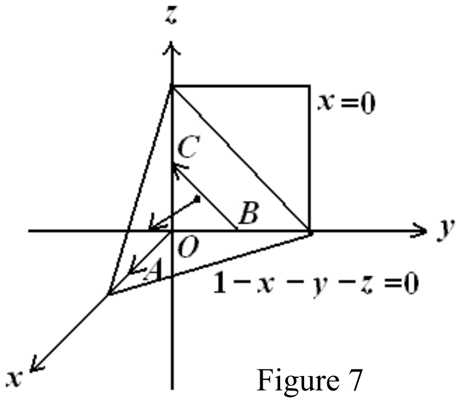


Figure 7

Let us calculate the minors:

$$\overline{BC}: \left.\begin{matrix} 1-x-y-z=0 \\ x=0 \end{matrix}\right\} \to \begin{vmatrix} -2 & 3 \\ -1 & 3 \end{vmatrix} = -3 \qquad -\overline{AC}: \left.\begin{matrix} 1-x-y-z=0 \\ y=0 \end{matrix}\right\} \to \begin{vmatrix} -2 & 3 \\ -1 & 10 \end{vmatrix} = -17$$

$$\overline{AB}: \left.\begin{matrix} 1-x-y-z=0 \\ z=0 \end{matrix}\right\} \to \begin{vmatrix} -2 & 3 \\ -2 & 4 \end{vmatrix} = -2 \qquad \overline{OC}: \left.\begin{matrix} x=0 \\ y=0 \end{matrix}\right\} \to \begin{vmatrix} -1 & 3 \\ -1 & 10 \end{vmatrix} = -7$$

$$-\overline{OB}: \left.\begin{matrix} x=0 \\ z=0 \end{matrix}\right\} \to \begin{vmatrix} -1 & 3 \\ -2 & 4 \end{vmatrix} = 2 \qquad \overline{OA}: \left.\begin{matrix} y=0 \\ z=0 \end{matrix}\right\} \to \begin{vmatrix} -1 & 10 \\ -2 & 4 \end{vmatrix} = 16$$

which yields the decomposition:

$$r = \pi_1 \cap \pi_2 = \begin{bmatrix} -2 \\ -1 \\ -1 \\ -2 \end{bmatrix} \wedge \begin{bmatrix} 3 \\ 3 \\ 10 \\ 4 \end{bmatrix} = 16\overline{OA} - 2\overline{OB} - 7\overline{OC} - 2\overline{AB} + 17\overline{AC} - 3\overline{BC}$$

Although the components are the same as those obtained by means of the exterior product of two points on the line, in general we expect to obtain only proportional components because they are homogeneous coordinates. When passing to barycentric coordinates we will have the same expression as before:

$$r = \frac{16}{19}\overline{OA} - \frac{2}{19}\overline{OB} - \frac{7}{19}\overline{OC} - \frac{2}{19}\overline{AB} + \frac{17}{19}\overline{AC} - \frac{3}{19}\overline{BC}$$

That is, the decomposition of a line for a basis of lines is unique, and the same result is obtained from two points on the line or from two planes whose intersection is the line. I have already proven this theorem in [5] from the matrix equation of the line.

Meet and join operators

Given two subspaces S_1 and S_2 of the point space E_n, the *meet* $\cap$ operator of the projective geometry [6] is defined as the set of points belonging simultaneously to both subspaces. For instance, $\pi_1 \cap \pi_2$ is the line intersection of both planes. Note that we have obtained this line from the exterior product of dual coordinates. In the same way, $\pi_1 \cap \pi_2 \cap \pi_3$ is the point intersection of the three planes, which can be obtained as the exterior product of their dual coordinates.

In order to make right calculations, the anticommutativity of the exterior product of points must be taken into account. How to work in an easy way with the *meet* operator? By cancellation of distinct points when the common points are in the same position in each factor:

$$\overline{OAB} \cap \overline{OBC} = -\overline{OBA} \cap \overline{OBC} = -\overline{OB}$$

$$\overline{OBC} \cap \overline{OAC} = \overline{OC}$$

$$\overline{OAB} \cap \overline{OAC} = \overline{OA}$$

$$\overline{OAB} \cap \overline{ABC} = \overline{ABO} \cap \overline{ABC} = \overline{AB}$$

$$\overline{OAC} \cap \overline{ABC} = -\overline{ACO} \cap \overline{ACB} = -\overline{AC}$$

$$\overline{OBC} \cap \overline{ABC} = \overline{BCO} \cap \overline{BCA} = \overline{BC}$$

In the same way:

$$\overline{OAB} \cap \overline{OAC} \cap \overline{OBC} = O$$

$$\overline{OAB} \cap \overline{OAC} \cap \overline{ABC} = \overline{AOB} \cap \overline{AOC} \cap \overline{ABC} = A$$

$$\overline{OAB} \cap \overline{OBC} \cap \overline{ABC} = \overline{BOA} \cap \overline{BOC} \cap \overline{BAC} = B$$

$$\overline{OAC} \cap \overline{OBC} \cap \overline{ABC} = \overline{COA} \cap \overline{COB} \cap \overline{CAB} = C$$

Let us see an example. Let us consider the point P given by the intersection of three planes:

$$P=\begin{cases} x+y-z=1 \\ 2x+y+z=3 \\ x+y-z=4 \end{cases} \qquad P=\begin{cases} (1-x-y-z)+2z=0 \\ 3\,(1-x-y-z)+x+2y+2z=0 \\ 4(1-x-y-z)+3x+3y+5z=0 \end{cases}$$

Since $\overline{OBC}$ has the equation $x=0$, $\overline{OAC}$ has the equation $-y=0$, $\overline{OAB}$ has the equation $z=0$ and $1-x-y-z$ is the equation of $-\overline{ABC}$ we have:

$$P=\begin{bmatrix}1\\0\\0\\2\end{bmatrix}\wedge\begin{bmatrix}3\\1\\2\\2\end{bmatrix}\wedge\begin{bmatrix}4\\3\\3\\5\end{bmatrix}$$

$$=\begin{vmatrix}1&3&4\\0&1&3\\0&2&3\end{vmatrix}\overline{ABC}\cap\overline{OBC}\cap\overline{OAC}-\begin{vmatrix}1&3&4\\0&1&3\\2&2&5\end{vmatrix}\overline{ABC}\cap\overline{OBC}\cap\overline{OAB}$$

$$+\begin{vmatrix}1&3&4\\0&2&3\\2&2&5\end{vmatrix}\overline{ABC}\cap\overline{OAC}\cap\overline{OAB}-\begin{vmatrix}0&1&3\\0&2&3\\2&2&5\end{vmatrix}\overline{OBC}\cap\overline{OAC}\cap\overline{OAB}$$

$$=\begin{vmatrix}1&3&4\\0&1&3\\0&2&3\end{vmatrix}C-\begin{vmatrix}1&3&4\\0&1&3\\2&2&5\end{vmatrix}B+\begin{vmatrix}1&3&4\\0&2&3\\2&2&5\end{vmatrix}A-\begin{vmatrix}0&1&3\\0&2&3\\2&2&5\end{vmatrix}O$$

$$P=-3C-9B+6A+6O \qquad P=2OA-3OB-OC=2e_1-3e_2-e_3$$

That is, P is a point at infinity located in the direction $2e_1-3e_2-e_3$, which is an expected result because two planes are parallel, and their intersection is a line at the infinity.

On the other hand, it has been shown that the exterior product of points generates lines and planes, so that it corresponds to the *join* operator $\cup$ of the projective geometry [6]. Summarizing the exterior product:

$$\wedge(\cdots)=\cup \qquad \wedge[\cdots]=\cap$$

Therefore Grassmann's exterior algebra reflects what is already known in projective geometry: the meet operator $\cap$ and the join operator $\cup$ are each one dual of each other.

Projectivities

The projective geometry only differs from the affine geometry in the fact that points at infinity are incorporated to the set of points, and transformations between points at infinity and points at a finite distance from the origin of coordinates are now admitted. Homogeneous coordinates allow us to work with points whose addition of barycentric coordinates is null, which are points at infinity.

From an algebraic point of view, a projectivity is just a non-degenerate linear mapping of the homogeneous coordinates:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \qquad \det \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \neq 0$$

Let us prove this assertion in two steps.

1) Firstly, let us prove that this mapping transforms lines into lines. Let P and Q be two points on a line. Then, their transformed points P' and Q' will be given by:

$$P' = \mathbf{M}\, P \qquad\qquad Q' = \mathbf{M}\, Q$$

where $\mathbf{M}$ is the matrix of the linear mapping (always with $\det \mathbf{M} \neq 0$). Any linear combination of P and Q is a point on the line $\overline{PQ}$:

$$R = \lambda P + \mu Q \qquad \Leftrightarrow \qquad R \in \overline{PQ}$$

The transformed point of R under the linear mapping is:

$$R' = \mathbf{M}\, R = \mathbf{M}\,(\lambda P + \mu Q) = \lambda\, \mathbf{M} P + \mu\, \mathbf{M} Q = \lambda\, P' + \mu\, Q'$$

Since R' is a linear combination of P' and Q', it always belongs to the line $\overline{P'Q'}$. Therefore, any line will always be transformed into another line. All these calculations are made with homogeneous coordinates, so that there is no need for $\lambda + \mu = 1$. The condition $\det \mathbf{M} \neq 0$ guarantees that any four independent points will be mapped into another set of four independent points, that is, a tetrahedron with a non null height will be transformed into another tetrahedron with a non null height. The condition $\det \mathbf{M} = 0$ corresponds to other collapsing geometric transformations that are not properly projectivities [7].

2) Finally, let us prove that this linear mapping

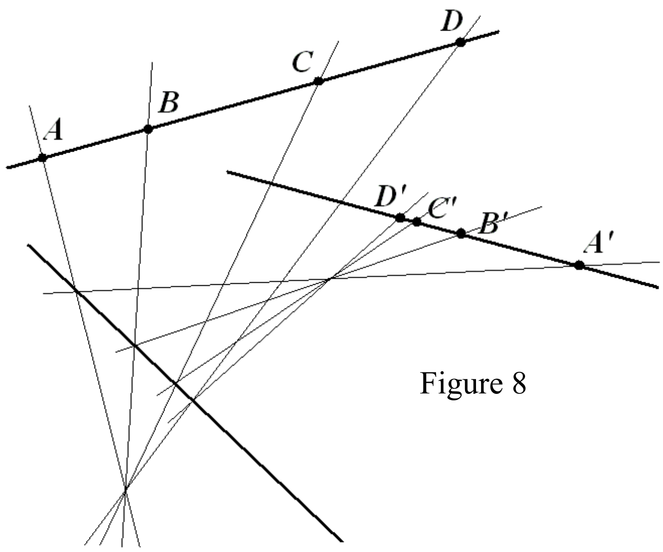


Figure 8

preserves the cross ratio. Let A, B, C and D be four distinct collinear points (figure 8). Then we can write C and D as barycentric linear combinations of A and B (barycentric coordinates are needed to measure distances on the line):

$$C = (1-\lambda)A + \lambda B \qquad D = (1-\mu)A + \mu B \qquad \lambda, \mu \in \mathbb{R} - \{0\}$$

Then we have:

$$AC = C - A = \lambda\, AB \qquad AD = D - A = \mu AB$$

$$BC = C - B = (\lambda - 1)AB \qquad BD = D - B = (\mu - 1)AB$$

The cross ratio is then:

$$\frac{AC\ BD}{AD\ BC} = \frac{\lambda\,(1-\mu)}{\mu\,(1-\lambda)}$$

Under the linear transformation, points A and B become A' and B':

$$A' = k\,\mathbf{M}A \qquad B' = l\,\mathbf{M}B \qquad k, l \in \mathbb{R} - \{0\}$$

where k and l are constants that are necessary to pass from the homogeneous coordinates of A' and B' to their barycentric coordinates. The points C and D are transformed into C' and D' in the same way:

$$C' = m\,\mathbf{M}C \qquad D' = n\,\mathbf{M}D \qquad m, n \in \mathbb{R} - \{0\}$$

The normalization coefficients m and n are related with k and l as now we see by substitution of their barycentric expression:

$$C' = m\,\mathbf{M}\left[(1-\lambda)A + \lambda B\right] = m\,(1-\lambda)\mathbf{M}A + m\lambda\,\mathbf{M}B = \frac{m\,(1-\lambda)}{k}A' + \frac{m\lambda}{l}B'$$

With barycentric coordinates, the addition of the coefficients of linear combination must be the unity:

$$\frac{m\,(1-\lambda)}{k} + \frac{m\lambda}{l} = 1 \qquad \Rightarrow \qquad m = \frac{kl}{k\lambda + l(1-\lambda)}$$

Therefore:

$$C' = \frac{l\,(1-\lambda)}{k\lambda + l(1-\lambda)}A' + \frac{k\lambda}{k\lambda + l(1-\lambda)}B'$$

Analogously:

$$D' = \frac{l\left(1-\mu\right)}{k\mu + l\left(1-\mu\right)} A' + \frac{k\mu}{k\mu + l\left(1-\mu\right)} B'$$

Then the vectors are:

$$A'C' = \frac{k\lambda}{k\lambda + l(1-\lambda)} A'B' \qquad A'D' = \frac{k\mu}{k\mu + l\left(1-\mu\right)} A'B'$$

$$B'C' = \frac{l\left(\lambda - 1\right)}{k\lambda + l(1-\lambda)} A'B' \qquad B'D' = \frac{\left(\mu - 1\right)}{k\mu + l\left(1-\mu\right)} A'B'$$

Now the cross ratio of the transformed points has the same value as that of the initial points:

$$\frac{A'C' \;\; B'D'}{A'D' \;\; B'C'} = \frac{\lambda\left(1-\mu\right)}{\mu\left(1-\lambda\right)} = \frac{AC \;\; BD}{AD \;\; BC}$$

Since the cross ratio of collinear points is preserved, this linear transformation is a projectivity and the proof ends.

A projectivity transforms collinear points into collinear points, which means that it must be a linear transformation of the homogeneous coordinates (in order to also include points at infinity). The most general way to write a linear transformation is with a matrix product. Therefore, it is not another way to write projectivities.

Quadrics

The general equation of a quadric is:

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + jz + k = 0$$

By introducing barycentric coordinates it becomes a quadratic form of them:

$$m_{11}\left(1-x-y-z\right)^2 + m_{22}x^2 + m_{33}y^2 + m_{44}z^2 + 2m_{12}\left(1-x-y-z\right)x$$

$$+ 2m_{13}\left(1-x-y-z\right)y + 2m_{14}\left(1-x-y-z\right)z + 2m_{23}xy + 2m_{24}xz + 2m_{34}yz = 0$$

In matrix form:

$$\begin{pmatrix} 1-x-y-z & x & y & z \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{12} & m_{22} & m_{23} & m_{24} \\ m_{13} & m_{23} & m_{33} & m_{34} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{pmatrix} \begin{pmatrix} 1-x-y-z \\ x \\ y \\ z \end{pmatrix} = 0$$

That is, a quadric is the kernel of a bilinear mapping. The matrix of a quadric is always symmetrical, and hence it can be passed to diagonal form:

$$\begin{pmatrix} 1-x'-y'-z' & x' & y' & z' \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}\begin{pmatrix} 1-x'-y'-z' \\ x' \\ y' \\ z' \end{pmatrix} =$$

$$= \lambda_1 (1-x'-y'-z')^2 + \lambda_2 (x')^2 + \lambda_3 (y')^2 + \lambda_4 (z')^2 = 0$$

with $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R} - \{0\}$. The change to diagonal form is obtained through a projectivity:

$$\begin{pmatrix} 1-x-y-z \\ x \\ y \\ z \end{pmatrix} = B \begin{pmatrix} 1-x'-y'-z' \\ x' \\ y' \\ z' \end{pmatrix} \qquad \det B \neq 0$$

The quadric only exists if there are eigenvalues with different signs. The quadric matrix is defined up to a non null factor: all the matrices of the form $\mathbf{M}' = k\mathbf{M}$ (with $k \in \mathbb{R} - \{0\}$) define the same quadric.

Tangential quadric

At each point of the quadric there is a plane tangent to it. Each of these planes is a point in the dual space, so that the set of all these planes form in the dual space the dual or tangential quadric. <u>The matrix of the dual quadric is always the inverse matrix of that of the original quadric</u> as I will show now. The equation of a quadric is:

$$\begin{pmatrix} 1-x-y-z & x & y & z \end{pmatrix} \mathbf{M} \begin{pmatrix} 1-x-y-z \\ x \\ y \\ z \end{pmatrix} = 0$$

By differentiation we obtain:

$$\begin{pmatrix} -\delta(x+y+z) & \delta x & \delta y & \delta z \end{pmatrix} \mathbf{M} \begin{pmatrix} 1-x-y-z \\ x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1-x-y-z & x & y & z \end{pmatrix} \mathbf{M} \begin{pmatrix} -\delta x - \delta y - \delta z \\ \delta x \\ \delta y \\ \delta z \end{pmatrix} = 0$$

By using the fact that the bilinear mapping is symmetrical, we can transpose matrices in the second term, which leads us to:

$$\begin{pmatrix} -\delta(x+y+z) & \delta x & \delta y & \delta z \end{pmatrix} \mathbf{M} \begin{pmatrix} 1-x-y-z \\ x \\ y \\ z \end{pmatrix} = 0$$

The plane being tangent to the quadric in the point (x_0, y_0, z_0) has the equation:

$$\begin{pmatrix} -(x-x_0+y-y_0+z-z_0) & x-x_0 & y-y_0 & z-z_0 \end{pmatrix} \mathbf{M} \begin{pmatrix} 1-x_0-y_0-z_0 \\ x_0 \\ y_0 \\ z_0 \end{pmatrix} = 0$$

that is:

$$\begin{pmatrix} 1-x-y-z & x & y & z \end{pmatrix} \mathbf{M} \begin{pmatrix} 1-x_0-y_0-z_0 \\ x_0 \\ y_0 \\ z_0 \end{pmatrix} = 0$$

because (x_0, y_0, z_0) is a point of the quadric and fulfils the equation:

$$\begin{pmatrix} 1-x_0-y_0-z_0 & x_0 & y_0 & z_0 \end{pmatrix} \mathbf{M} \begin{pmatrix} 1-x_0-y_0-z_0 \\ x_0 \\ y_0 \\ z_0 \end{pmatrix} = 0$$

The dual coordinates of this plane are the coefficients of the barycentric coordinates in the equation of the plane. Let t', u', v' y w' be these homogeneous dual coordinates:

$$\begin{bmatrix} t' \\ u' \\ v' \\ w' \end{bmatrix} = \mathbf{M} \begin{pmatrix} 1-x_0-y_0-z_0 \\ x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

Let **X** be the matrix of the quadric points and **U** the matrix of the dual (homogeneous or normalized) coordinates:

$$\mathbf{X} = \begin{pmatrix} 1-x-y-x \\ x \\ y \\ z \end{pmatrix} \qquad \mathbf{U} = \begin{pmatrix} t' \\ u' \\ v' \\ w' \end{pmatrix}$$

Then, we have:

$$\mathbf{U} = \mathbf{M}\,\mathbf{X}$$

$$\mathbf{M}^{-1}\,\mathbf{U} = \mathbf{X}$$

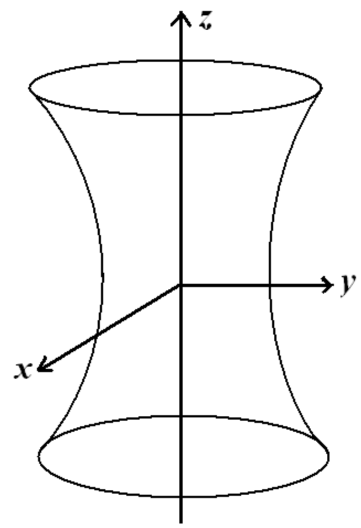
$$\mathbf{X}^{\mathrm{T}} = \mathbf{U}^{\mathrm{T}}\,\mathbf{M}^{-1}$$

because $\mathbf{M}^{-1}$ as well as $\mathbf{M}$ are symmetrical and do not change under transposition. The substitution of $\mathbf{X}$ in the equation of the quadric $\mathbf{X}^{\mathrm{T}}\,\mathbf{M}\,\mathbf{X} = 0$ gives us:

$$\mathbf{U}^{\mathrm{T}}\,\mathbf{M}^{-1}\,\mathbf{U} = 0$$

So that the proof ends.

Let us see an example. Let us consider now a one-sheeted hyperboloid with the z-axis as its revolution axis (figure 9):



$$x^2 + y^2 - z^2 = 1$$

Figure 9

$$2x^2 + 2y^2 - (1 - x - y - z)^2 - 2x - 2y - 2z + 2xy + 2xz + 2yz = 0$$

$$-2z^2 - (1 - x - y - z)^2 - 2(1 - x - y - z)(x + y + z) - 2xy - 2xz - 2yz = 0$$

$$2z^2 + (1 - x - y - z)^2 + 2(1 - x - y - z)(x + y + z) + 2xy + 2xz + 2yz = 0$$

$$\begin{pmatrix} 1-x-y-z & x & y & z \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1-x-y-z \\ x \\ y \\ z \end{pmatrix} = 0$$

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} \quad \Rightarrow \quad \mathbf{M}^{-1} = \begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

So that the equation of the tangential quadric is:

$$\begin{bmatrix}1-u-v-w & u & v & w\end{bmatrix}\begin{bmatrix}0 & 1 & 1 & -1\\ 1 & -1 & 0 & 0\\ 1 & 0 & -1 & 0\\ -1 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1-u-v-w\\ u\\ v\\ w\end{bmatrix}=0$$

$$-2u^2-2v^2+4w^2-2uv+2uw+2vw-4w+1=0$$

where $[u, v, w]$ are the dual coordinates of any plane tangent to the point quadric. For instance, let us calculate the plane tangent to the hyperboloid at the point $(x, y, z)=(1, 1, 1)$. By differentiation of the equation of the hyperboloid we find:

$$2x\,dx+2y\,dy-2z\,dz=0$$

At the point $(1, 1, 1)$ we have:

$$dx+dy-dz=0$$

Taking finite differences, the equation of the tangent plane π is:

$$\pi:\quad (x-1)+(y-1)-(z-1)=0 \qquad \Leftrightarrow \qquad x+y-z-1=0$$

Let us introduce barycentric coordinates:

$$\pi:\quad (1-x-y-z)+2z=0 \qquad \Leftrightarrow \qquad \pi=\begin{bmatrix}1 & 0 & 0 & 2\end{bmatrix}$$

and let us check that it is a point of the tangential quadric:

$$\begin{bmatrix}1 & 0 & 0 & 2\end{bmatrix}\begin{bmatrix}0 & 1 & 1 & -1\\ 1 & -1 & 0 & 0\\ 1 & 0 & -1 & 0\\ -1 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\ 0\\ 0\\ 2\end{bmatrix}=0 \quad ✓$$

The tangential quadric only exists if $\det\mathbf{M}\neq 0$, which corresponds to proper quadrics. For $\det\mathbf{M}=0$ the inverse matrix of $\mathbf{M}$ does not exist. This case corresponds to degenerate quadrics, such as a pair of planes or a cone. For instance, the degenerate quadric:

$$(x+y-2z)(1-x+y)=0 \quad \Rightarrow \quad x+y-2z=0 \quad \text{or} \quad 1+x+y=0$$

represents two planes cutting on a line. By developing the Cartesian equation we have

$$-x^2+y^2+2xz-2yz+x+y-2z=0$$

Let us write it in barycentric coordinates:

$$-x^2+y^2+2xz-2yz+(x+y-2z)(1-x-y-z)+(x+y-2z)(x+y+z)=0$$

$$2y^2-2z^2+2xy+xz-3yz+x(1-x-y-z)+y(1-x-y-z)-2z(1-x-y-z)=0$$

$$\begin{pmatrix}1-x-y-z & x & y & z\end{pmatrix}\begin{pmatrix}0 & 1/2 & 1/2 & -1\\ 1/2 & 0 & 1 & 1/2\\ 1/2 & 1 & 2 & -3/2\\ -1 & 1/2 & -3/2 & -2\end{pmatrix}\begin{pmatrix}1-x-y-z\\ x\\ y\\ z\end{pmatrix}=0$$

We se that its determinant is null:

$$\det\begin{pmatrix}0 & 1/2 & 1/2 & -1\\ 1/2 & 0 & 1 & 1/2\\ 1/2 & 1 & 2 & -3/2\\ -1 & 1/2 & -3/2 & -2\end{pmatrix}=0$$

and therefore the tangential quadric does not exist.

Measure theory of extensions

The exterior product of points generate, as we have seen, lines, planes and the whole space. However, it is also used to obtain the measure of the extensions among these points. According to Peano's methodology, very well explained in [8], a linear operator ω which maps products of *k* points into products of $k-1$ vectors is defined in the following way:

$$\omega:(E_n)^k \to (\wedge V_n)^{k-1}$$

$$\omega(1)=0$$
$$\omega(P)=1$$
$$\omega\left(\overline{P_0P_1}\right)=P_1-P_0=P_0P_1$$
$$\omega\left(\overline{P_0P_1P_2}\right)=(P_1-P_0)\wedge(P_2-P_0)=P_0P_1\wedge P_0P_2$$
$$\cdots$$
$$\omega\left(\overline{P_0P_1\cdots P_n}\right)=(P_1-P_0)\wedge(P_2-P_0)\wedge\cdots\wedge(P_n-P_0)=P_0P_1\wedge\cdots\wedge P_0P_n$$

In fact, as we will now see, it is enough to state $\omega\left(\overline{P_0P_1}\right)=P_1-P_0$ because linearity yields the rest of equalities.

For instance, let us see how the exterior product of two points is. Given two points P_1 and P_2 given with barycentric coordinates:

$$P_1 = (1 - x_1 - y_1 - z_1)O + x_1 A + y_1 B + z_1 C$$

$$P_2 = (1 - x_2 - y_2 - z_2)O + x_2 A + y_2 B + z_2 C$$

their product is:

$$P_1P_2 = \begin{vmatrix} 1 - x_1 - y_1 - z_1 & x_1 \\ 1 - x_2 - y_2 - z_2 & x_2 \end{vmatrix} OA + \begin{vmatrix} 1 - x_1 - y_1 - z_1 & y_1 \\ 1 - x_2 - y_2 - z_2 & y_2 \end{vmatrix} OB + \begin{vmatrix} 1 - x_1 - y_1 - z_1 & z_1 \\ 1 - x_2 - y_2 - z_2 & z_2 \end{vmatrix} OC$$

$$+ \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} AB + + \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} AC + \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} BC$$

Now, by considering the vectorial equalities $AB = OB - OA$, $AC = OC - OA$ and $BC = OC - OB$ we have:

$$\omega\left(\overline{P_1P_2}\right) = P_1P_2 = \begin{vmatrix} 1 - x_1 & x_1 \\ 1 - x_2 & x_2 \end{vmatrix} (A - O) + \begin{vmatrix} 1 - y_1 & y_1 \\ 1 - y_2 & y_2 \end{vmatrix} (B - O) + \begin{vmatrix} 1 - z_1 & z_1 \\ 1 - z_2 & z_2 \end{vmatrix} (C - O)$$

$$= (x_2 - x_1)\,OA + (y_2 - y_1)OB + (z_2 - z_1)OC$$

which is the usual expression of a segment given by two points in Cartesian coordinates. For our students we would write:

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\,\overrightarrow{OA} + (y_2 - y_1)\overrightarrow{OB} + (z_2 - z_1)\overrightarrow{OC}$$

Therefore, the exterior product of two points given with barycentric coordinates generates the line element not only in direction but also in extension, that is, it generates a segment.

Let us see the exterior product of three points:

$$P_1P_2P_3 = \begin{vmatrix} 1 - x_1 - y_1 - z_1 & x_1 & y_1 \\ 1 - x_2 - y_2 - z_2 & x_2 & y_2 \\ 1 - x_3 - y_3 - z_3 & x_3 & y_3 \end{vmatrix} OAB + \begin{vmatrix} 1 - x_1 - y_1 - z_1 & x_1 & z_1 \\ 1 - x_2 - y_2 - z_2 & x_2 & z_2 \\ 1 - x_3 - y_3 - z_3 & x_3 & z_3 \end{vmatrix} OAC$$

$$+ \begin{vmatrix} 1 - x_1 - y_1 - z_1 & y_1 & z_1 \\ 1 - x_2 - y_2 - z_2 & y_2 & z_2 \\ 1 - x_3 - y_3 - z_3 & y_3 & z_3 \end{vmatrix} OBC + \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} ABC$$

Since the oriented areas fulfil $ABC = OAB + OBC + OCA$ [9] we have:

$$P_1P_2P_3 = \begin{vmatrix} 1 - x_1 - y_1 & x_1 & y_1 \\ 1 - x_2 - y_2 & x_2 & y_2 \\ 1 - x_3 - y_3 & x_3 & y_3 \end{vmatrix} OAB + \begin{vmatrix} 1 - x_1 - z_1 & x_1 & z_1 \\ 1 - x_2 - z_2 & x_2 & z_2 \\ 1 - x_3 - z_3 & x_3 & z_3 \end{vmatrix} OAC + \begin{vmatrix} 1 - y_1 - z_1 & y_1 & z_1 \\ 1 - y_2 - z_2 & y_2 & z_2 \\ 1 - y_3 - z_3 & y_3 & z_3 \end{vmatrix} OBC$$

$$= \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} OAB + \begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \end{vmatrix} OAC + \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{vmatrix} OBC$$

After subtracting the first row from the second and third rows in the determinants, we obtain:

$$P_1P_2P_3 = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} OAB + \begin{vmatrix} x_2 - x_1 & z_2 - z_1 \\ x_3 - x_1 & z_3 - z_1 \end{vmatrix} OAC + \begin{vmatrix} y_2 - y_1 & z_2 - z_1 \\ y_3 - y_1 & z_3 - z_1 \end{vmatrix} OBC$$

which expresses an exterior product of vectors:

$$P_1P_2P_3 = P_1P_2 \wedge P_1P_3$$

The triangle area equals to half this exterior product.

And finally the exterior product of four points given in barycentric coordinates is:

$$P_1P_2P_3P_4 = \begin{vmatrix} 1 - x_1 - y_1 - z_1 & x_1 & y_1 & z_1 \\ 1 - x_2 - y_2 - z_2 & x_2 & y_2 & z_2 \\ 1 - x_3 - y_3 - z_3 & x_3 & y_3 & z_3 \\ 1 - x_4 - y_4 - z_4 & x_4 & y_4 & z_4 \end{vmatrix} OABC$$

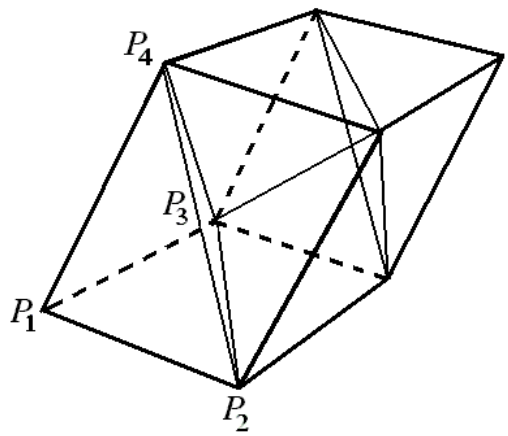


Figure 10

which we can write as:

$$P_1P_2P_3P_4 = \begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix} OABC = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} OABC$$

which expresses the volume of the parallelepiped as the exterior product of three of its non parallel edges (figure 10):

$$P_1P_2P_3P_4 = P_1P_2 \wedge P_1P_3 \wedge P_1P_4$$

The volume of the tetrahedron is 1/6 of the volume of this exterior product.

Grassmann's extension theory is fully general and independent of the Euclidean or pseudo Euclidean character of a given hyperspace of n dimension. In the same way, Möbius' barycentric calculus and the barycentric coordinates use a basis of any points, not necessarily located on perpendicular coordinate axes. For a simplex with n vertices [10], its extension is given by:

$$\text{Simplex}_{P_1P_2\cdots P_n} = \frac{1}{(n-1)!} P_1P_2 \cdots P_n$$

where $P_1P_2 \cdots P_n$ expresses the exterior product of points given with barycentric coordinates. If the affine space containing these points has vectorial dimension lower than $n-1$, the points are always linear dependent and their exterior product is null. Thus, there exist simplexs with n vertices in spaces with vectorial dimension $m \geq n-1$. For instance, in the room space $m=3$ and we can have segments ($n=2$), triangles $(n=3)$ and tetrahedrons $(n=4)$. In the space-time we have besides hypertetrahedrons with five vertices ($n=5$):

$$\text{Hypertetrahedron}_{P_1P_2P_3P_4P_5} = \frac{1}{24} P_1P_2P_3P_4P_5$$

Its hypervolume is calculated by means of 1/24 of the determinant of its barycentric space-time coordinates.

Acknowledgments

I wish to thank my pupils of these last 25 years for their patience and understanding during my lessons on linear algebra and geometry, from which the ideas for this talk come. My particular vision, based on the barycentric coordinates and the exterior product, often differs from textbooks and official curriculum, but I'm sure that it has provided them with a wider and more logical comprehension of mathematics.

Thank you very much for listening to me and I will reply to your questions if possible.

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