Why and how the geometric algebra should be taught at high school. Experiences and proposals.

Ramon González Calvet
I.E.S. Pere Calders, Cerdanyola del Vallès
Spain

The research as a source of new pedagogic methods of mathematics

There is a close dependence between the pedagogic methods and the research lines. In fact, there exists a continuous interchange between new advances in science and new methods of teaching it. The actual lines of research will become the new subjects to teach. I may give some examples: the differential calculus, the algebra, the arithmetic, etc.

First example: the differential calculus

Newton and Leibniz discovered the differential (and integral) calculus, which was also developed by others researchers such as L'Hôpital, Bernoulli and Euler. In the XVII and XVIII centuries the differential calculus was an active field of research. In the XIX century it becomes an academic subject at the university level. In the XX century the Newton fluxions (derivatives) and the Leibniz integrals and differentials are taught to 16-17 years old pupils at high schools. Aside of the main relation between the derivative and the primitive, I enumerate other theorems taught at these levels:

The chain rule for derivatives of composed functions.
The Barrow's rule for the definite integral.
The Rolle's theorem.
The mean value theorem.
The Cauchy's theorem.
The L'Hôpital rule to calculate indeterminate limits.

Second example: the linear algebra

In the field of the linear algebra, recall that the matrices were discovered by Arthur Cayley (*A memoir on the theory of matrices*, 1858) although Leibniz already worked with the determinants. Nowadays, foundations of linear algebra are taught at high schools:

The operations with matrices
The determinants.
Solving the systems of linear equations with determinants.
The Gauss method to solve systems and invert matrices.
The Rouché-Frobenius theorem.

Third example: the Ancients' geometry

Also we have other examples. Archimedes discovered that the volumes of the cone, sphere and cylinder having the same diameter and altitude are in the ratio 1:2:3. This aphorism was engraved in his tomb. Geometry was an active field of research during the Hellenistic period. Now we are teaching the formulas of the volumes to 12-14 years old pupils but also other geometric theorems such as:

Theorems about angles.
Theorem of the isosceles triangle.
Formulas to calculate the area of planar figures.
Formulas to calculate volumes.

Fourth example: the arithmetic

Also the field of arithmetic has been incorporated to the curricula of high schools. Among others we teach (12 years old pupils):

The fundamental theorem of arithmetic (decomposition of a number in prime factors).
The maximum common divisor and the minimum common multiple.
Divisibility criteria.
Fractions (introduced by Egyptians).
Decimal fractions (introduced by Simon Stevin in 1585).

In algebra, nowadays we use the algebraic notation mainly developed by Viète.

As a last example, I teach in the subject of computer science the binary numbers, which were firstly worked by Leibniz.

So we must expect that any new field of mathematics, now upon research, in a little time will become an academic subject, also at high school. We may remember the Ernst Haeckel's aphorism: "the ontogeny recapitulates the phylogeny". Although this statement is actually considered a falsehood, we may adapt its sense and say:

"<u>Teaching mathematics recapitulates its history</u>"

The field of the geometric algebra will also follow this path. Those new areas now under the scope of research will be soon learned by our pupils also at high school. In fact the main problem of the geometric algebra is the blockade it suffers mainly at the university (at least in Spain, although we have the feeling that this is a general situation in all the world). So our contributions to the geometric algebra will soon become an academic subject if we are able to overcome this blockade.

However any coin has two faces and this is also applicable to the interaction between pedagogy and research. My experience teaching mathematics (16 years) has shown me that the process of teaching is a constant source of new ideas and inspiration for those that are working in research. My lessons on geometry, analysis and linear algebra at high school have supplied me many new ideas for advance in the research of geometric algebra. I may affirm that the *Treatise of plane geometry through geometric algebra* would not exist if I were not teacher of mathematics because of the lack of the source of inspiration.

The title of the conference is:

"How and why the geometric algebra should be taught at high school. Experiences and proposals."

because we must accelerate the collapse of the barriers preventing that the geometric algebra be reckoned an important academic subject, a first and main front being the high school. At this level, the mind of our pupils is very flexible and receptive. Any seed of geometric algebra sown at this age will give very successful fruits whenever there exists continuity of the teaching at the university level. The main question is not the difficulty of the geometric algebra for our pupils, since any topic may be adapted to the corresponding level. The true problem is the inertia of the departments (of the teachers themselves), which tend to impart the same subjects that have been always taught. However do not think that these subjects are a good sample of all the mathematics. Not at all, but usually they are a partial and slanted view of them. So the renovation, the up-to-date reintroduction of geometry through the geometric algebra will be a hard task and likely we shall have to battle against our own colleagues.

Now I give some examples about how we may teach parts of the geometric algebra to our students. I'm mainly considering 15-16 aged pupils, but also you may lengthen this range to first courses of university. The transition from high school to college should be more gradual than the current one. In our school we intend that pupils do not suffer a crack in mathematics during this transition and I believe that we are achieving in some degree this purpose. However, if we introduce concepts of geometric algebra at high school and they do not have continuity at university, what is the yield of this effort?

Barycentric coordinates

One of the most interesting approaches to point geometry is the use of barycentric coordinates. Any generic point on a line may be written as linear combination of two given points of this line:

$$R = (1-k)\,P + k\,Q$$

with coefficients whose addition is the unity. These coefficients are the barycentric coordinates. In the same way, the linear combination of two lines is the pencil of lines passing through the intersection of both lines. Any line of the pencil has the following general equation:

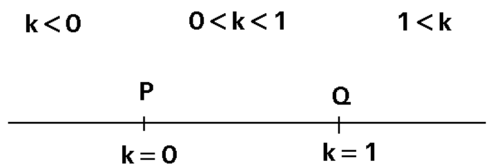


Figure 1

$$(1-p)\left(n_1 x + n_2 y + c\right) + p\left(n_1 x + n_2 y + c'\right) = 0$$

In the space, we may write any plane of a pencil of planes for a given line as linear combination of two planes of this pencil. Consider the problem of finding the plane that contains a line and an outer point. If the line is given as intersection of two planes, the best way to solve this problem is to use the pencil of planes containing this line. Example:

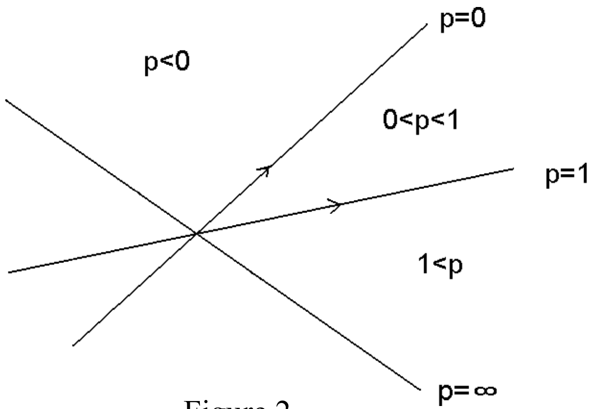


Figure 2

Find the plane containing the line

$$\begin{cases} 2x+3y=4 \\ x+y-z=2 \end{cases}$$ and the point (3, 4, –2).

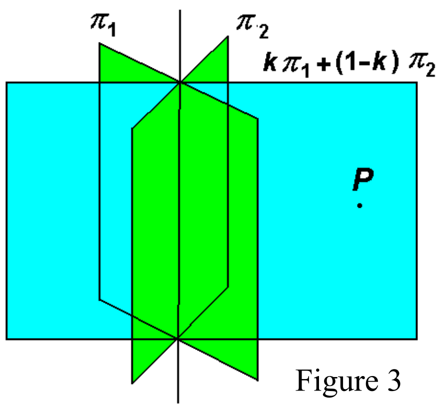


Figure 3

Take the pencil of planes for this line. Any plane of this pencil has the equation (using barycentric coordinates):

$$k\left(2x+3y-4\right)+\left(1-k\right)\left(x+y-z-2\right)=0$$

but it must contain the point:

$$k\left(2\cdot 3+3\cdot 4-4\right)+\left(1-k\right)\left(3+4+2-2\right)=0 \qquad \Rightarrow \qquad k=-1$$

and so the searched plane is: $y+2z=0$

In fact, a system of Cartesian coordinates hides the barycentric coordinates of the points (figure 4):

$$R=\left(x,\,y\right)=O+x\,OP+y\,OQ\,=\left(1-x-y\right)O+x\,P+y\,Q$$

That is, any point R can be written as linear combination of three non-aligned points $\{O,P,Q\}$ with barycentric coordinates always summing the unity.

An immediate application of the barycentric coordinates is the calculus of the oriented area S of a triangle ABC:

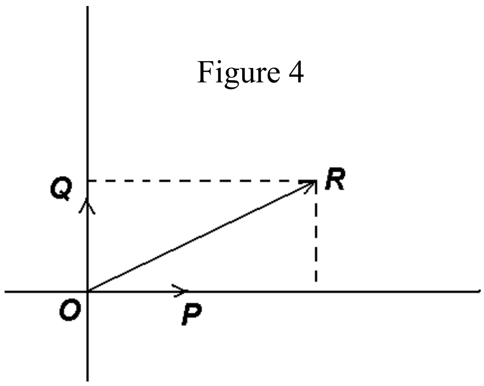


Figure 4

$$S_{ABC}=\frac{1}{2}AB\wedge BC=\frac{1}{2}\begin{vmatrix} 1-x_A-y_A & x_A & y_A \\ 1-x_B-y_B & x_B & y_B \\ 1-x_C-y_C & x_C & y_C \end{vmatrix} e_{12}$$

which vanishes if the three points are aligned.

In the same way, the oriented volume of the tetrahedron having vertexes *A, B, C* and *D* may be written using barycentric coordinates:

$$V = \frac{1}{6} AB \wedge BC \wedge CD = \frac{1}{6} \begin{vmatrix} 1 - x_A - y_A - z_A & x_A & y_A & z_A \\ 1 - x_B - y_B - z_B & x_B & y_B & z_B \\ 1 - x_C - y_C - z_C & x_C & y_C & z_C \\ 1 - x_D - y_D - z_D & x_D & y_D & z_D \end{vmatrix} e_{123}$$

The barycentric coordinates are very useful in projective geometry. A projectivity is simply defined as a linear transformation of the barycentric coordinates:

$$\begin{pmatrix} 1 - x' - y' \\ x' \\ y' \end{pmatrix} = k \begin{pmatrix} h_{00} & h_{01} & h_{02} \\ h_{10} & h_{11} & h_{12} \\ h_{20} & h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} 1 - x - y \\ x \\ y \end{pmatrix} \qquad \det h_{ij} \neq 0$$

where k is a variable number allowing that the transformed barycentric coordinates have sum equal to the unity.

Dual coordinates in the dual plane

The concept of pencil of lines leads to the definition of the dual plane, whose barycentric coordinates are those of the pencils of lines. Given a point base $\{O, P, Q\}$ the dual base is formed by the lines $\overline{OP}$, $\overline{PQ}$ and $\overline{QO}$. For Cartesian coordinates, the dual base are $-x - y + 1 = 0$, $x = 0$ and $y = 0$. Any line on the plane is expressed as a linear combination of these three lines using barycentric coordinates. Let us calculate the dual Cartesian coordinates of the line $2x + 3y + 4 = 0$. We must solve the identity:

$$2x + 3y + 4 \equiv a'(-x - y + 1) + b'x + c'y \qquad \forall\, x, y$$

$$x(2 + a' - b') + y(3 + a' - c') + 4 - a' \equiv 0$$

whose solution is:

$$a' = 4 \qquad b' = 6 \qquad c' = 7$$

Dividing by the sum of the coefficients we obtain:

$$\frac{2x + 3y + 4}{17} \equiv \frac{4}{17}(-x - y + 1) + \frac{6}{17}x + \frac{7}{17}y$$

whence the dual coordinates of this line are obtained as $[b, c] = [6/17, 7/17]$. Let us see their meaning. The linear combination of both coordinates axes is a line of the pencil of lines passing through the origin:

$$\frac{6}{13}x + \frac{7}{13}y = 0 \qquad \text{or} \qquad 6x + 7y = 0$$

This line intersects the third base line $-x - y + 1 = 0$ at the point (7, –6), whose pencil of lines is described by:

$$a\left(-x-y+1\right)+\left(1-a\right)\left(\frac{6}{13}x+\frac{7}{13}y\right)=0$$

Then $2x + 3y + 4 = 0$ is the line of this pencil determined by $a = 4/17$.

Three lines are concurrent if the determinant of their barycentric dual coordinates vanishes. On the other hand, the line at the infinity has dual coordinates [1/3, 1/3], that is , it is the barycenter of the dual base.

Also the equation of any conic has very simple form using barycentric coordinates:

$$\begin{pmatrix}1-x-y & x & y\end{pmatrix}\mathbf{S}\begin{pmatrix}1-x-y\\ x\\ y\end{pmatrix}=0$$

where **S** is a symmetric matrix. I have shown that the matrix of the tangential conic (the conic plotted in the dual plane by the dual points corresponding to the tangents to the point conic) is equal to the inverse of the matrix of the point conic, its equation being:

$$\begin{bmatrix}1-a-b & a & b\end{bmatrix}\mathbf{S}^{-1}\begin{bmatrix}1-a-b\\ a\\ b\end{bmatrix}=0$$

where $[a, b]$ are the Cartesian coordinates in the dual plane.

Scalar and exterior product in the plane

The metric geometry explained to our pupils is a skew geometry, that is, a censured geometry. Only those problems involving scalar product are considered. However the scalar and exterior products arise in a very symmetric form in geometry. We may and must teach to our pupils both products:

$$v\cdot w=v_x w_x+v_y w_y \qquad\qquad v\wedge w=\left(v_x w_y-v_y w_x\right)e_{12}$$

Then the geometric product may be introduced with the help of complex numbers:

$$v\,w=v\cdot w+v\wedge w$$

A slight version may be to use a real instead of imaginary exterior product. Anyway it allows pupils to calculate the areas of triangles without having its altitude.

Plane trigonometry and the hyperbolic plane

Many years ago I had prepared this table where the identities of the circular and hyperbolic functions are compared. But then I could not imagine in which extent the geometric algebra develops the analogy between circular and hyperbolic trigonometry.

TRIGONOMETRIA 1-

CIRCULAR	HIPERBOLICA
$\sin^2 x + \cos^2 x = 1$	$ch^2 x - sh^2 x = 1$
$tg\, x = \frac{\sin x}{\cos x}$	$th\, x = \frac{sh\, x}{ch\, x}$
$\sin(-x) = -\sin x$	$sh(-x) = -sh\, x$
$\cos(-x) = \cos x$	$ch(-x) = ch\, x$
$tg(-x) = -tg\, x$	$th(-x) = -th\, x$
SUMA D'ANGLES	
$\sin(x+y) = \sin x \cos y + \cos x \sin y$	$sh(x+y) = sh\, x\, ch\, y + ch\, x\, sh\, y$
$\cos(x+y) = \cos x \cos y - \sin x \sin y$	$ch(x+y) = ch\, x\, ch\, y + sh\, x\, sh\, y$
$tg(x+y) = \frac{tg\, x + tg\, y}{1 - tg\, x\, tg\, y}$	$th(x+y) = \frac{th\, x + th\, y}{1 + th\, x\, th\, y}$
ANGLES DOBLES	
$\sin 2x = 2 \sin x \cos x$	$sh\, 2x = 2\, sh\, x\, ch\, x$
$\cos 2x = \cos^2 x - \sin^2 x$	$ch\, 2x = ch^2 x + sh^2 x$
$tg\, 2x = \frac{2\, tg\, x}{1 - tg^2 x}$	$th\, 2x = \frac{2\, th\, x}{1 + th^2 x}$
ANGLES MEITAT	
$\sin \frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$	$sh \frac{x}{2} = \pm\sqrt{\frac{ch\, x - 1}{2}}$ + pr $x>0$, − pr $x<0$
$\cos \frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$	$ch \frac{x}{2} = \sqrt{\frac{ch\, x + 1}{2}}$
$tg \frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$	$th \frac{x}{2} = \pm\sqrt{\frac{ch\, x - 1}{ch\, x + 1}} = \frac{ch\, x - 1}{sh\, x} = \frac{sh\, x}{ch\, x + 1}$
SUMA I RESTA DE FUNCIONS	
$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$	$sh\, x + sh\, y = 2\, sh \frac{x+y}{2}\, ch \frac{x-y}{2}$
$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$	$sh\, x - sh\, y = 2\, sh \frac{x-y}{2}\, ch \frac{x+y}{2}$
$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$	$ch\, x + ch\, y = 2\, ch \frac{x+y}{2}\, ch \frac{x-y}{2}$
$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$	$ch\, x - ch\, y = 2\, sh \frac{x+y}{2}\, sh \frac{x-y}{2}$
$tg\, x \pm tg\, y = \frac{\sin(x \pm y)}{\cos x \cos y}$	$th\, x \pm th\, y = \frac{sh(x \pm y)}{ch\, x\, ch\, y}$

Figure 5

Like the complex numbers are those naturally associated with the circular trigonometry, the hyperbolic numbers are those naturally associated with the hyperbolic trigonometry:

$$z = a + b\,e_{12} \quad e_{12}^2 = -1 \qquad\qquad z = a + b\,e_1 \quad e_1^2 = 1$$

So we have the analogous of Euler's and De Moivre's identities:

$$\exp(x\,e_{12}) = \cos x + e_{12}\sin x \qquad \exp(x\,e_1) = \cosh x + e_1 \sinh x$$

$$(\cos x + e_{12}\sin x)^n = \cos nx + e_{12}\sin nx \qquad (\cosh x + e_1\sinh x)^n = \cosh nx + e_1 \sinh nx$$

An Euclidean angle is defined as the quotient of the arc length divided by the radius of the circle:

$$\alpha = \frac{s}{r} = \frac{2A}{r^2}$$

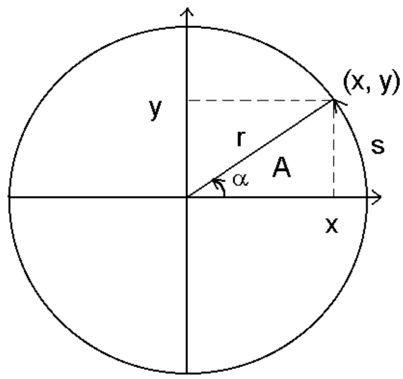


Figure 6

and it is proportional to the area of the circular sector.

In the same way a hyperbolic angle is defined as the quotient of the arc length of hyperbola divided by the radius of the hyperbola in the pseudo-Euclidean plane:

$$\psi = \frac{s}{r} = \frac{2A}{r^2}$$

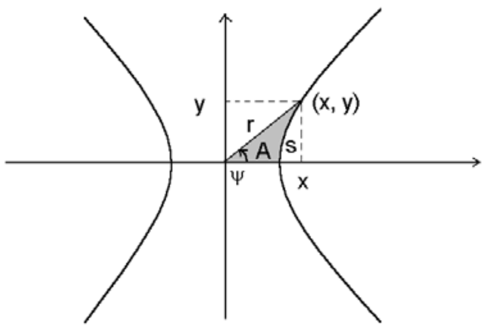


Figure 7

Of course, we cannot represent the hyperbolic arguments with circle arcs as made in the plane trigonometry. So a new sketch of the hyperbolic arguments by means of arcs of hyperbola is needed. In the figure 8 we see a typical hyperbolic triangle suitably drawn to show that the addition of the three angles is $-\pi\, e_{12}$:

The relation between angles in different quadrants are shown in the figure 9:

Figure 9

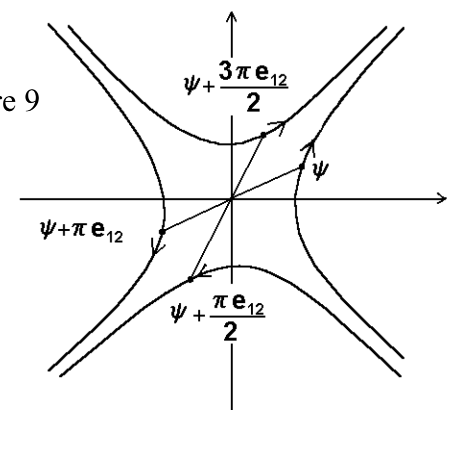
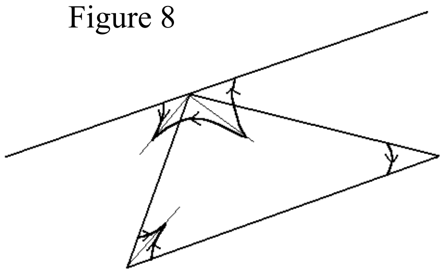


Figure 8



Also the laws of sines, cosines and tangents for any generic Euclidean triangle:

$$\frac{|a|}{\sin\alpha} = \frac{|b|}{\sin\beta} = \frac{|c|}{\sin\gamma}$$

$$a^2 = b^2 + c^2 - 2\,|b|\,|c|\cos\alpha$$

$$\frac{|a|+|b|}{|a|-|b|} = \frac{\tan\dfrac{\alpha+\beta}{2}}{\tan\dfrac{\alpha-\beta}{2}}$$

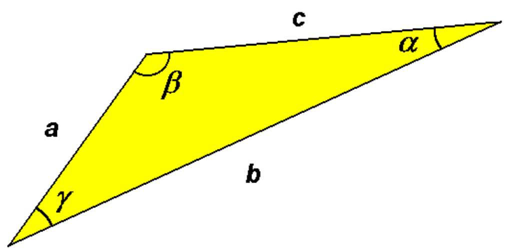


Figure 10

have their analogous laws for hyperbolic triangles:

$$\frac{|a|}{\sinh\alpha} = \frac{|b|}{\sinh\beta} = \frac{|c|}{\sinh\gamma}$$

$$a^2 = b^2 + c^2 - 2\,|b|\,|c|\cosh\alpha$$

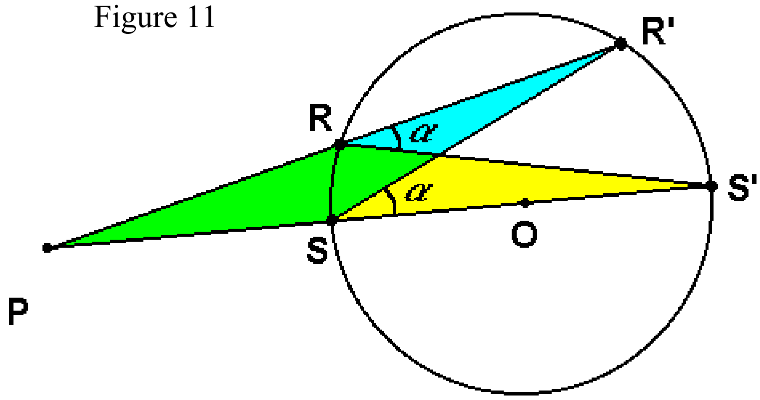
$$\frac{|a|+|b|}{|a|-|b|} = \frac{\tanh\dfrac{\alpha+\beta}{2}}{\tanh\dfrac{\alpha-\beta}{2}}$$

Some examples and proofs showed me that they work with the suitable orientation of the hyperbolic arguments displayed in the figure 9.

We also explain to our pupils the power of a point with regard to a circle, which is the value obtained when the coordinates of the point are substituted in the equation of the circle.

$$PR\;\; PR' = PS\;\; PS' = PO^2 - OS^2 = (x - x_0)^2 + (y - y_0)^2 - r^2$$

Figure 11



In the hyperbolic plane, the power of a point with regard to a hyperbola is also constant. The proof is as follows: the yellow and blue hyperbolic triangles in the figure are similar because the hyperbolic argument ψ (in the hyperbolic plane) is constant and equal to a half of the arc length *R'S'* of the hyperbola. The opposite similarity may be written using the geometric product:

$$PR\,(PS')^{-1} = (PR')^{-1}\,PS$$

what implies:

$$PR'\,PR = PS\;\;PS' = PO^2 - OS^2 = (x - x_0)^2 - (y - y_0)^2 - r^2$$

Observe that the power of a point with regard to a hyperbola in the hyperbolic plane is just obtained using the Cartesian equation! This clearly shows that Cartesian coordinates do not necessarily mean Euclidean coordinates. This extreme was already criticized by Leibniz.

On the other hand, I took this figure for the cover of my *Treatise of plane geometry through geometric algebra*.

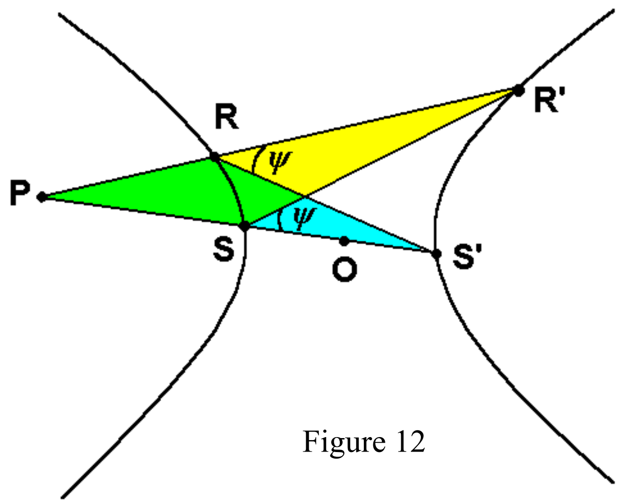


Figure 12

Rank of a matrix, exterior product and systems of equations

We explain to our pupils that the rank of a matrix is the number of linearly independent rows or columns. In the method that uses determinants to find the rank, we take an element of the matrix and we add rows and columns taking always that matrix having non-null determinant:

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 3 & 1 & 3 & 2 \\ -3 & 1 & 0 & 5 \end{pmatrix}$$

$$1 \neq 0 \rightarrow \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} = -2 \neq 0 \rightarrow \begin{vmatrix} 1 & 1 & 2 \\ 3 & 1 & 2 \\ -3 & 1 & 0 \end{vmatrix} = 0 \rightarrow \begin{vmatrix} 1 & 1 & 3 \\ 3 & 1 & 2 \\ -3 & 1 & 5 \end{vmatrix} = 0 \quad \Rightarrow \quad \text{rank}\,\mathbf{M} = 2$$

If all the determinants obtained adding one row and one column to a non-null n-dimensional determinant are zero then all the other n+1 dimensional determinants are null. This ensures that the rank of the matrix is equal to n.

In the alternative method using exterior product we have:

$$(e_1 + e_2 + 2e_3 + 3e_4) \wedge (3e_1 + e_2 + 3e_3 + 2e_4) =$$

$$= -2e_1 \wedge e_2 - 3e_1 \wedge e_3 - 7e_1 \wedge e_4 + e_2 \wedge e_3 - e_2 \wedge e_4 - 5e_3 \wedge e_4$$

The coefficients of the exterior product are the corresponding minors of the matrix. Now we make the exterior product with the third vector:

$$(e_1 + e_2 + 2e_3 + 3e_4) \wedge (3e_1 + e_2 + 3e_3 + 2e_4) \wedge (-3e_1 + e_2 + 5e_4) = 0$$

So we conclude that the rank of the matrix **M** is 2.

Note that the determinants of a given order n, which are the components of the exterior product of n vectors are not linearly independent although they are orthogonal from the point of view of the Pythagorean theorem. Solely I wish to manifest that the following words are not synonyms in geometric algebra: *linearly independent* ≠ *orthogonal* ≠ *perpendicular*. I may explain more about this question at the end of the talk if you wish.

Another application of the exterior product is the resolution of systems of equations.

$$\begin{cases} a_{11}x_1 + \quad \cdots \quad + a_{1n}x_n = b_1 \\ \quad \vdots \qquad\qquad\qquad \vdots \\ a_{n1}x_1 + \quad \cdots \quad + a_{nn}x_n = b_n \end{cases}$$

which may be written as a vectorial equality:

$$x_1 v_1 + \cdots + x_n v_n = b$$

where $v_i = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$. In order to solve it, we take the exterior product with all the other vectors:

$$x_i \; v_1 \wedge v_2 \wedge \cdots \wedge v_n = v_1 \wedge \cdots \wedge v_{i-1} \wedge b \wedge v_{i+1} \wedge \cdots \wedge v_n$$

whence it follows the Cramer's rule:

$$x_i = \frac{v_1 \wedge \cdots \wedge v_{i-1} \wedge b \wedge v_{i+1} \wedge \cdots \wedge v_n}{v_1 \wedge v_2 \wedge \cdots \wedge v_n} = \frac{\det(v_1 \cdots v_{i-1} \; b \; v_{i+1} \cdots v_n)}{\det(v_1 \; v_2 \cdots v_n)}$$

Change of coordinates

One set of coordinates may be easily changed to another set of coordinates with the exterior product. For example, the change from Cartesian to spherical coordinates. We have:

$$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$$

$$dx = \sin\theta\cos\varphi\; dr + r\cos\theta\cos\varphi\; d\theta - r\sin\theta\sin\varphi\; d\varphi$$

$$dy = \sin\theta\sin\varphi\; dr + r\cos\theta\sin\varphi\; d\theta + r\sin\theta\cos\varphi\; d\varphi$$

$$dz = \cos\theta\; dr - r\sin\theta\; d\theta$$

When taking the exterior product of the three differentials we obtain the volume element:

$$dV = dx\wedge dy\wedge dz = r^2\sin\theta\; dr\wedge d\theta\wedge d\varphi$$

But we can apply it to any geometric element such us the surface element:

$$dx\wedge dy = r\sin^2\theta\; dr\wedge d\varphi + r^2\sin\theta\cos\theta\; d\theta\wedge d\varphi$$

$$dy\wedge dz = -r\sin\varphi\; dr\wedge d\theta + r^2\sin^2\theta\cos\varphi\; d\theta\wedge d\varphi + r\sin\theta\cos\theta\cos\varphi\; d\varphi\wedge dr$$

$$dz\wedge dx = r\cos\varphi\; dr\wedge d\theta + r^2\sin^2\theta\sin\varphi\; d\theta\wedge d\varphi + r\sin\theta\cos\theta\sin\varphi\; d\varphi\wedge dr$$

$$dA = dx\wedge dy + dy\wedge dz + dz\wedge dx$$

This is a vectorial equation whence one deduces the modulus of the element of area:

$$|\,dA\,|^2 = |\,dx\wedge dy\,|^2 + |\,dy\wedge dz\,|^2 + |\,dz\wedge dx\,|^2 =$$

$$= r^2|\,dr\wedge d\theta\,|^2 + r^4\sin^2\theta\,|\,d\theta\wedge d\varphi\,|^2 + r^2\sin^2\theta\,|\,d\varphi\wedge dr\,|^2$$

So we have the differential of area in spherical coordinates:

$$dA = r\; dr\wedge d\theta + r^2\sin\theta\; d\theta\wedge d\varphi + r\sin\theta\; d\varphi\wedge dr$$

Rotations and symmetries

The axial symmetry (or reflection) shown in the figure may be expressed in the form:

$$PR' = v^{-1}PR\; v$$

where v is the direction vector of the axis of symmetry. This is easily proved because the perpendicular component anticommutes with the vector v:

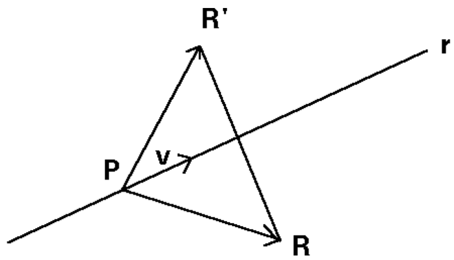
$$v^{-1}\left(PR_{\perp} + PR_{\parallel}\right)v = v^{-1}v\left(-PR_{\perp} + PR_{\parallel}\right) = -PR_{\perp} + PR_{\parallel}$$

If the point R belongs to the line, the vector PR and the direction vector v are proportional and commute:

$$PR\, v = v\, PR \quad \Leftrightarrow \quad PR\, v - v\, PR = 0 \quad \Leftrightarrow \quad PR \wedge v = 0 \quad \Leftrightarrow \quad PR = v^{-1}\, PR\, v$$

The last equation is the *algebraic equation* of a line and shows that the vector PR remains invariant under a reflection in the direction of the line. That is, the point R only belongs to the line when it coincides with the point reflected in this line. Separating components we have:

Figure 13



$$\frac{x - x_P}{v_1} = \frac{y - y_p}{v_2}$$

Also, a rotation of angle α may be expressed in a way analogous to axial symmetries:

$$v' = v\left(\cos\alpha + e_{12}\sin\alpha\right) = \left(\cos\alpha/2 - e_{12}\sin\alpha/2\right)v\left(\cos\alpha/2 + e_{12}\sin\alpha/2\right)$$

The first form is only valid on a plane, while the second form is general for any dimension. If we take any complex number with argument $\alpha/2$ we may write:

$$v' = z^{-1} v\, z \qquad\qquad z = |z|_{\alpha/2}$$

Now we may easily prove which transformation is the composition of two reflections with regard to directions v and w:

$$t' = v^{-1} t\, v$$

$$t'' = w^{-1} t'\, w = w^{-1} v^{-1} t\, v\, w = z^{-1} v\, z$$

$$z = v\, w$$

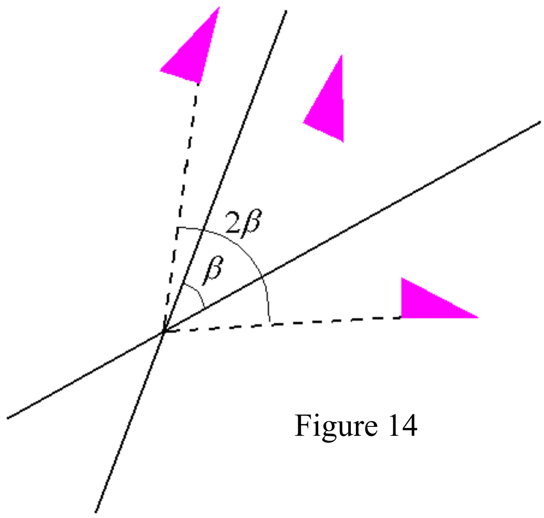


Figure 14

The product of two vectors is a complex number and therefore the composition of two reflections is a rotation over a double of the angle between both directions. If the directions are parallel, the rotation has center at the infinity and becomes a translation along a double distance of that between the axes of symmetry.

Of course, the composition of two rotations on the plane is another rotation over an angle equal to the addition of both angles as follows from the product of complex numbers in polar form:

$$z\,t = |z|_{\alpha}\,|\,t\,|_{\beta} = |z|\,|\,t\,|_{\alpha+\beta}$$

In the Euclidean space, each rotation is described in the following way:

$$v' = (\cos\alpha/2 - u\sin\alpha/2)\,v\,(\cos\alpha/2 + u\sin\alpha/2)$$

where u is a unitary bivector. In general:

$$v' = q^{-1} v\, q$$

where q is any quaternion whose plane is the plane of rotation and whose argument is the half of the rotation angle. This expression is applied to any element of the algebra of the three-dimensional space such as scalars, vectors, bivectors or volumes. However in the technological applications the use of only bivectors and quaternions is preferred.

The composition of two rotations in any planes is obtained through the product of quaternions:

$$v'' = r^{-1} q^{-1} v\, q\, r$$

This is the best way to describe rotations and composition of rotations with immediate technological interest, such as the engineers have discovered long time ago.

However there is a difficulty with the mirror reflections, which cannot be written in this way. This has a physical consequence: an asymmetric molecule cannot be converted into its mirror image, as Louis Pasteur showed in the case of tartaric acid.

Notable points of a triangle

The conditions of intersection of the medians, bisectors of the sides, the angle bisectors and the altitudes lead to geometric equations for the notable points of a triangle PQR whose solutions are respectively:

$$G = \frac{P+Q+R}{3}$$ (centroid)

$$O = -(P^2 QR + Q^2 RP + R^2 PQ)(2\,PQ \wedge QR)^{-1}$$ (circumcenter)

$$I = \frac{P\,|QR| + Q\,|RP| + R\,|PQ|}{|QR| + |RP| + |PQ|}$$ (incenter)

$$H = (P\;\; P\cdot QR + Q\;\; Q\cdot RP + R\;\; R\cdot PQ)(QR \wedge RP)^{-1}$$ (orthocenter)

$$N = (P\,QR\,P + Q\,RP\,Q + R\,PQ\,R)(4\,PQ \wedge QR)^{-1}$$ (center of the nine-point circle)

Observe that the points lying on the Euler line have expressions implying the geometric (Clifford) product (for the centroid is not needed but we may also add a factor and a divisor $(PQ \wedge QR)^{-1}$), expressions which cannot be written using only the scalar product of the so called *metric geometry*. This shows clearly the importance of the geometric product in geometry and the partial vision and censure with which the geometry is nowadays taught.

Some comments on the Lobachevskian geometry

Also the Lobachevsky's geometry may be easily explained to our pupils using geometric algebra. The two sheeted hyperboloid with constant radius in the pseudo-Euclidean space realizes the Lobachevsky's geometry:

$$z^2 - x^2 - y^2 = 1$$

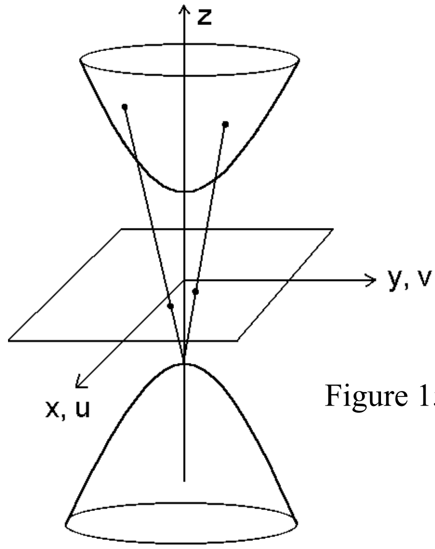
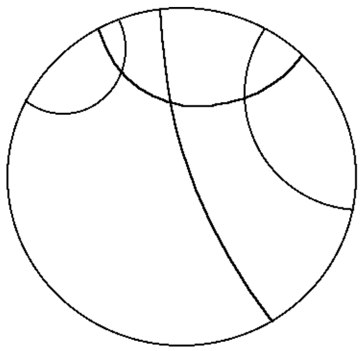


Figure 15

Starting from this point we may deduce expressions of the change of coordinates for any projection.

Let us see, for instance, the stereographic projection or Poincaré disk with more detail. This conformal projection whose point of view is the pole of one sheet is displayed by the figures 15 and 16:

Figure 16



$$\frac{x}{u} = z + 1 \qquad \frac{y}{v} = z + 1$$

where u, v are the coordinates on the plane of projection. The arc length is:

$$ds = \sqrt{dx^2 + dy^2 - dz^2} = \frac{2\sqrt{du^2 + dv^2}}{1 - u^2 - v^2}$$

The exterior product allows us to calculate quickly the differential of area:

$$dA = \sqrt{(dx \wedge dy)^2 - (dy \wedge dz)^2 - (dz \wedge dx)^2} = \frac{4\, du \wedge dv}{(1 - u^2 - v^2)^2}$$

Also with the help of the exterior product we may deduce the azimuthal equivalent projection, which preserves areas, by imposing that the area differentials in Cartesian coordinates and in the projection must be equal:

$$\begin{cases} dA^2 = \frac{1}{z^2}(dx \wedge dy)^2 = (du \wedge dv)^2 \\ u = x\ f(z) \qquad v = y\ f(z) \end{cases}$$

$$u = x\sqrt{\frac{2}{z+1}} \qquad\qquad v = y\sqrt{\frac{2}{z+1}}$$

Other projections of the hyperboloid are the central projection (Beltrami's disk), whose point of view is the origin of coordinates, the cylindrical equidistant projection, which uses the Weierstrass coordinates analogous of the spherical coordinates:

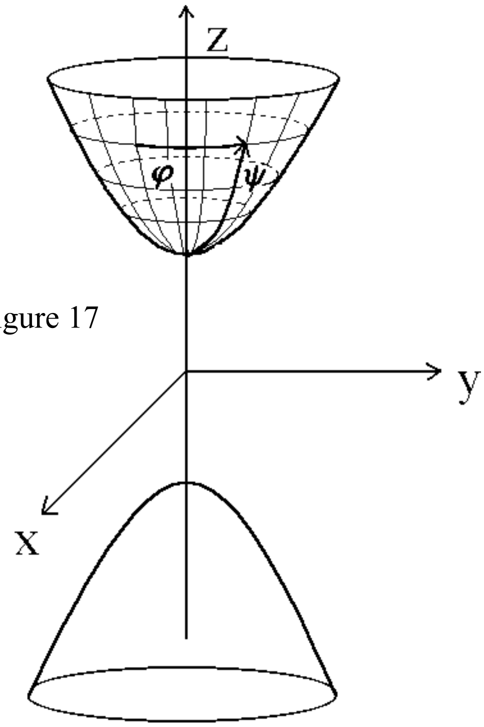


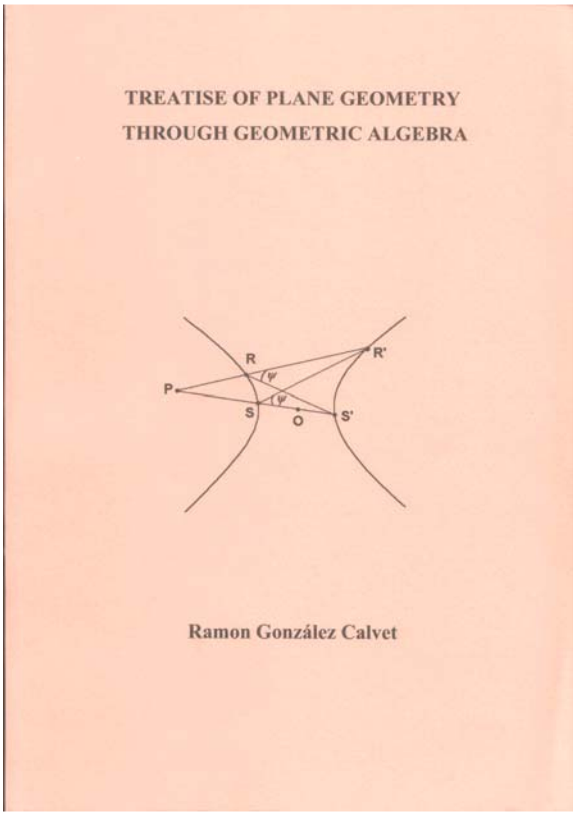
Figure 17

$$x = \sinh\psi \cos\varphi$$
$$y = \sinh\psi \sin\varphi$$
$$z = \cosh\psi$$

$$ds^2 = d\psi^2 + \sinh^2\psi\ d\varphi^2$$

$$dA = \sinh\psi\ d\psi \wedge d\varphi$$

the cylindrical conformal projection (analogous of Mercator's projection) and the conic equidistant and conformal projections. For all of them the exterior product allows us to calculate the area differential



The *Treatise of plane geometry through geometric algebra*

As a pedagogic tool for the introduction of geometric algebra, you have available the *Treatise of plane geometry through geometric algebra*. This book has four parts:

1. The vector plane and the complex numbers
2. The geometry of the Euclidean plane.
3. Pseudo-Euclidean geometry.
4. Plane projections of tri-dimensional spaces.

and many solved exercises in each lesson. It may be used as a reference book for high school pupils, but also as a textbook for introductory courses on geometric algebra or preliminary courses on plane geometry at the first university year. In fact my aim in writing it was to be a bridge between high school and university and a help for teachers. In this book are collected a significant part of the lessons given in the summer courses on geometric algebra for teachers that we (Josep Manel Parra and me) have imparted in the framework of the *Escola d'Estiu de Secundària* of the Col·legi de Doctors i Llicenciats en Filosofia i Lletres i Ciències de Catalunya. Perhaps you believe that I'm selling the book but in any case I do not make business since you may freely download it from Internet at the address:

http://campus.uab.es/~PC00018

Thank you very much for listening to me and I will answer your questions if possible.