Estimate of the error committed in determining the place from where a photograph has been taken

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1 Introduction

The article [3] deals with the thus called "problem of the photograph", which can be stated in the following way: if a minimum of five buildings whose site on the map of a city is known can be identified in a photograph of that city, it is requested to locate on the map the site of the point from where the photograph was taken. We will summarize firstly the procedure described in [3] to solve this problem.

We designate with A, B, C, D and E the sites of the five identified buildings on the map of the city. We trace a horizontal straight line on the photograph and we designate with A', B ', C', D' and E' the intersections of this horizontal straight line with the verticals of the buildings corresponding to A, B, C, D and E on the photograph (figure 1). Measuring on the photograph, we calculate two cross ratios r = (A', B', C', D') and s = (A', B', C', E'). If X is any point of the map, we designate with (XA, XB, XC, XD) the cross ratio of the four straight lines $\overline{XA}$, $\overline{XB}$, $\overline{XC}$, $\overline{XD}$. According to the Chasles theorem the geometric place of the points X such that (XA, XB, XC, XD) = r is a conic section passing through the points A, B, C and D. The view point of the photograph is the intersection point of both conics:

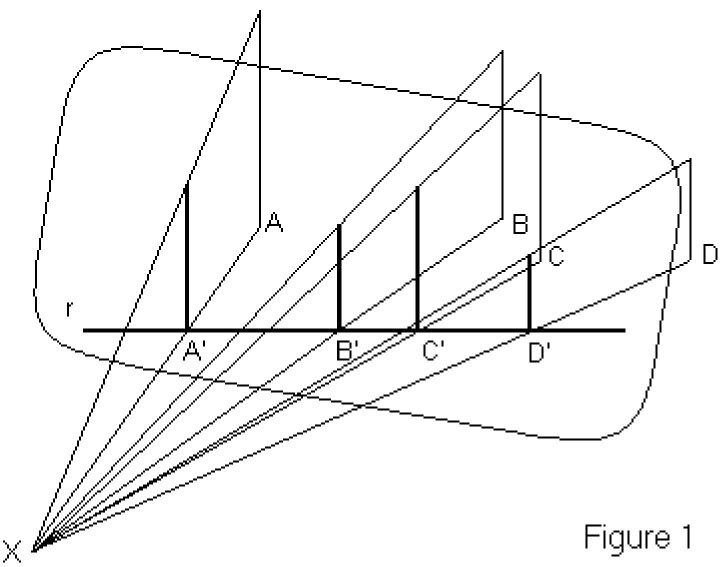

Figure 1

(1) $$\begin{cases} r = (XA, XB, XC\ XD) \\ s = (XA, XB, XC, XE) \end{cases}$$

which is neither A nor B nor C.

We will provide a procedure to calculate the cross ratio of four straight lines $\overline{XA}$, $\overline{XB}$, $\overline{XC}$, $\overline{XD}$. From the theorem 1 of [3]:

(2)

$$(XA, XB, XC, XD) = \frac{\sin \widehat{AXC}\ \sin \widehat{BXD}}{\sin \widehat{AXD}\ \sin \widehat{BXC}}$$

To obtain a most agreeable formula of the cross ratio, we defined the outer product of two vectors u, v in the plane in the following way (Grassmann, p. 99):

$$\wedge: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

(3) $u, v \rightarrow u \wedge v = u_x v_y - u_y v_x = |u||v| \sin(u, v)$

Designating with XA the vector A - X from X to A, we obtain:

(4) $XA \wedge XC = |XA||XC| \sin \widehat{AXC}$, etc.

From here we obtain by substitution into the former expression (2) of (XA, XB, XC, XD):

(5) $(XA, XB, XC, XD) = \dfrac{(XA \wedge XC)(XB \wedge XD)}{(XA \wedge XD)(XB \wedge XC)}$

Because of this, the conics whose intersection gives the view point of the photograph have the equations:

(6) $r = \dfrac{(XA \wedge XC)(XB \wedge XD)}{(XA \wedge XD)(XB \wedge XC)}$ $s = \dfrac{(XA \wedge XC)(XB \wedge XD)}{(XA \wedge XD)(XB \wedge XC)}$

From now on, to simplify the notation, we will designate with A, B, C, D and E the points on the map where the buildings that we have identified are located, as well as the corresponding buildings on the photograph.

In the calculation there is a great quantity of data, 10 coordinates for the map and 5 for the photograph, and the previous tests have shown that this method accumulates errors easily. Thus, we have thought that we must analyze the error factors and try to quantify their magnitude with the aim of minimizing them.

2 Error factors

The objective of this work is not only to determine the credibility range of the calculated coordinates, but also to establish the best calculation conditions which minimize the errors.

The factors which can produce errors are the following:

a) To suppose that the photograph has been taken in a vertical plane, when it is really slightly oblique. If the photograph is taken in a vertical plane, the verticals are projected on the photograph as parallel straight lines, but if the photograph lies on an oblique plane, the verticals are projected in the plane of the photograph as straight lines concurrent in a point. If the angle between the vertical and the plane of the photograph is small, the concurrence point is quite far from the center of the photograph and the projections can be considered parallel (figure 1). On the other hand, if the photograph has much obliquity (for example, the photograph of the belfry of a church taken closely), then one must measure the cross ratio of the pencil of projections of the verticals. For the two analyzed examples (with a slope of - 3ºand - 2º approximately) the corners of the different buildings are observed as parallel lines, with a precision of one pixel of the digitized images.

b) Imprecision in the determination of the vertical or horizontal reference. The fact that the base of the photograph is not horizontal (figure 1) has no importance if one has this reference (for example, a building, a communication tower, the sea horizon, etc.). Otherwise (for example in the photograph of a landscape), some error will be introduced because the cross ratio is calculated from the photograph on a plane which differs from the horizontal. In this case one must use a level associated with the photographic camera. When the cross ratio is measured by hand on the photograph, the best method is to trace the positions of the well identified objects on a millimetric tracing paper, which allows to align well the vertical lines of the millimetric paper with high buildings or other references. If the photograph has been digitized with a scanner, it can be turned using an adequate program until the image is placed perfectly horizontal whenever one has a reference.

c) Distortion ought to photographic optics. The photographic image is usually formed in a plane and it is approximately a flat projection of space. The distortion is the difference between the real photographic image and the expected flat projection (Sirohi-Kothiyal, p. 86). That the image is formed with the minimal distortion is the principle which guides the construction of most of the photographic cameras in the market. Some special objectives, such as the inverted telephoto lenses, have a great distortion, which disables a reliable calculation. The usual objectives such as those with focal distance of 50 mm do not present any significant distortion. The center of the perspective of a negative of 24 x 36 mm obtained through these objectives is located at 50 mm from the negative. For an enlargement of 9 x 13 cm, the center of the perspective, from where we should watch the photograph to have a vision like the real one with the same vision angles, is located at 18 cm (Malacara, p. 80) from the paper.

d) Errors in the distances measured in the photograph. A good method to reduce them is to digitize the photograph and to determine the coordinates on the computer screen. In this case one must turn the image until it is thoroughly horizontal. The errors of the measure by hand can be significant if we take into account that there are a lot of measures and that they are not very accurate (usual error of 1 mm on 10 cm).

e) Errors of measure or location of the points in the plane. Proper measure errors of the coordinates can be minimized taking an enlargement of the map. Nevertheless, sometimes it is hard to determine the exact situation in the plane of the constructions or objects which we observe; for example, when observing a tower whose center is located inside a block. Furthermore, the maps are not always good enough representations of reality. A small variation in the location of a point on the map with regard to reality can yield significant errors in the result. If one has a program containing the map (as *VisualMap* which has the map of Barcelona), the coordinates can be measured with much precision.

f) A map, as flat representation of a spherical surface, can not preserve simultaneously angles and distances (Strahler, p. 24-25). The smaller the area represented in the map is, the smaller will be the error when supposing that the map is an exact representation of reality and the buildings are located in a plane and not on a spherical surface.

Below we will study the propagation of measure errors, the most significant ones, in the calculation of the position of point X from where the photograph has been taken.

3 Differential technique

To evaluate how the initial errors influence the calculated position of the point in the plane from where the photograph has been taken, we will use the d or differential technique. For each variable X which we measure, we have its error dX, a random variable centered at zero. The square root of the variance is taken as a measure of the error range:

(7) $X = E(X) + dX$

(8) $E(dX) = E(X - E(X)) = 0$

(9) $Var(dX) = Var(X - E(X)) = Var(X) = E(dX^2)$

The differential technique is suitable when the relative errors of any random variable X are small enough to use the Taylor series with only the first derivative. For this case, the function of the error of X is proportional to its derivative calculated at the value of X, considered constant (Dunn, appendix 3, p. 171-174). If the function depends on several variables we have:

(10) $$d f[X_i] = \Sigma_i \frac{\partial f}{\partial X_i} d X_i$$

(11) $$Var(f[X_i]) = \Sigma_i \left(\frac{\partial f}{\partial X_i}\right)^2 Var(X_i) + \Sigma_{i\neq}\Sigma_k \left(\frac{\partial f}{\partial X_i}\right)\left(\frac{\partial f}{\partial X_k}\right) Cov(X_i, X_k)$$

where the partial derivatives are calculated at the measured values of X_i. If all variables X_i are independent, then all covariances are null:

(12) $$Var(f[X_i]) = \Sigma_i \left(\frac{\partial f}{\partial X_i}\right)^2 Var(X_i)$$

4 Errors analysis for the photograph.

Firstly, we analyze the error propagated through the calculation of the cross ratio from the photograph. The cross ratio of the vertical projections A, B, C and D of the elements identified in the photograph is:

(13) $$r = (A\,B\,C\,D) = \frac{AC\;BD}{BC\;AD}$$

Differentiating, we obtain:

(14) $$\frac{dr}{r} = \frac{d\,AC}{AC} + \frac{d\,BD}{BD} - \frac{d\,BC}{BC} - \frac{d\,AD}{AD}$$

Taking into account that AC = C - A, it is found:

(15) $$\frac{dr}{r} = dA\left(-\frac{1}{AC} + \frac{1}{AD}\right) + dB\left(-\frac{1}{BD} + \frac{1}{BC}\right) + dC\left(\frac{1}{AC} - \frac{1}{BC}\right)$$
$$+ dD\left(\frac{1}{BD} - \frac{1}{AD}\right) = dA\,\frac{CD}{AC\,AD} - dB\,\frac{CD}{BC\,BD} + dC\,\frac{AB}{AC\,BC} - dD\,\frac{AB}{AD\,BD}$$

The four differential are statistically independent if we measure A, B, C, D from an arbitrary origin O. In this case:

(16) $$\frac{Var(r)}{r^2} = Var(A)\,\frac{CD^2}{AC^2\,AD^2} + Var(B)\,\frac{CD^2}{BC^2\,BD^2} + Var(C)\frac{AB^2}{AC^2\,BC^2} + Var(D)\frac{AB^2}{AD^2\,BD^2}$$

If the cross ratio is not very small, its error can be minimized when making small the distances AB and CD and large the others AC, BC, AD and BD. When the points A, B, C, D are ordered from left to right, this means that we take the points in the following way:

A B C D

For an extreme case, both AB and CD are null and the cross ratio becomes the unity.

It means that A and B and the observation point are aligned in the plane, as well as C and D and the observation point. In this case, the location in the plane is trivial. One traces the straight line going through A and B, and that one going through C and D. The intersection of both straight lines is the point from where the photograph was taken. This case is infrequent because it is difficult to recognize places with the same vertical in the photograph.

Generally, intuition indicates it is convenient that the points A, B, C, D are separate enough to make the relative errors in the measures of the distances be sufficiently small. But it is not evident the convenience that some points are nearer than others.

Similarly, for the cross ratio of the points A, B, C, D is obtained:

(17) $$s = (A\,B\,C\,E) = \frac{AC\;BE}{BC\;AE}$$

(18) $$\frac{Var(s)}{s^2} = Var(A)\,\frac{CE^2}{AC^2\,AE^2} + Var(B)\,\frac{CE^2}{BC^2\,BE^2} + Var(C)\,\frac{AB^2}{AC^2\,BC^2} + Var(E)\,\frac{AB^2}{AE^2\,BE^2}$$

following that an analogous disposition for the four points A, B, C and E minimizes the error:

A B C E

The variables r and s are not independent because their covariance is not null:

(19) $$Cov(r, s) = E(r\,s) - E(r)\,E(s) = E([E(r) + d\,r]\,[E(s) + ds]) - E(r)\,E(s)$$

$$= E(r)\,E(ds) + E(s)\,E(dr) + E(dr\,ds) = E(dr\,ds)$$

Considering that A, B, C, D, E are independent variables, we have:

(20) $E(dA\,dB) = E(dA)\,E(dB) = 0 \qquad E(dA\,dC) = 0$, etc.

(21) $$\frac{Cov(r,s)}{r\,s} = Var(A)\frac{CD\;CE}{AC^2\;AD\;AE} + Var(B)\frac{CD\;CE}{BC^2\;BD\;BE} + Var(C)\frac{AB^2}{AC^2\;BC^2}$$

If r and s are not null, the covariance vanishes when AB and CD or CE become zero:

A B D C E

A B E C D

If, furthermore ,we want the variances of r and s to be small or null, we must take the configuration:

A B C D E

It would seem that it is not convenient to take the points D and E closely, but it must be pointed out that one only has to do this in the photograph because in the plane they are not neighboring points. That is, these points are aligned in two view lines.

5 Error analysis for the plane

To make the error calculation for the map points one takes the logarithms of r and s and calculates their differentials:

(22) $$\frac{dr}{r} = dXA \wedge \left(\frac{XC}{XA \wedge XC} - \frac{XD}{XA \wedge XD}\right) + dXB \wedge \left(\frac{XD}{XB \wedge XD} - \frac{XC}{XB \wedge XC}\right) +$$
$$+ dXC \wedge \left(-\frac{XA}{XA \wedge XC} + \frac{XB}{XB \wedge XC}\right) + dXD \wedge \left(-\frac{XB}{XB \wedge XD} + \frac{XA}{XA \wedge XD}\right)$$

Separating the differentials of the point coordinates and taking into account for the coefficient of dX that XC - XA = AC, etc., it is obtained:

(23) $$\frac{dr}{r} = dX \wedge \left(-\frac{AC}{XA \wedge XC} + \frac{AD}{XA \wedge XD} - \frac{BD}{XB \wedge XD} + \frac{BC}{XB \wedge XC}\right) +$$
$$+ dA \wedge \left(\frac{XC}{XA \wedge XC} - \frac{XD}{XA \wedge XD}\right) + dB \wedge \left(\frac{XD}{XB \wedge XD} - \frac{XC}{XB \wedge XC}\right) +$$
$$+ dC \wedge \left(-\frac{XA}{XA \wedge XC} + \frac{XB}{XB \wedge XC}\right) + dD \wedge \left(-\frac{XB}{XB \wedge XD} + \frac{XA}{XA \wedge XD}\right)$$

Similarly, the differential of the cross ratio of the points A, B, C and E is:

(24) $$\frac{ds}{s} = dX \wedge \left(-\frac{AC}{XA \wedge XC} + \frac{AE}{XA \wedge XE} - \frac{BE}{XB \wedge XE} + \frac{BC}{XB \wedge XC}\right) +$$
$$+ dA \wedge \left(\frac{XC}{XA \wedge XC} - \frac{XE}{XA \wedge XE}\right) + dB \wedge \left(\frac{XE}{XB \wedge XE} - \frac{XC}{XB \wedge XC}\right) +$$

$$+ dC \wedge \left(- \frac{XA}{XA \wedge XC} + \frac{XB}{XB \wedge XC} \right) + dE \wedge \left(- \frac{XB}{XB \wedge XE} + \frac{XA}{XA \wedge XE} \right)$$

In order to understand better the factors of the differentials, we modify the coefficients of dX using the fact that the outer product of two vectors with the same direction is zero:

(25) $$XA \wedge XC = XA \wedge (XA + AC) = XA \wedge AC$$

$$XA \wedge XD = XA \wedge (XA + AD) = XA \wedge AD$$

$$XB \wedge XC = XB \wedge (XB + BC) = XB \wedge BC$$

$$XB \wedge XD = XB \wedge (XB + BD) = XB \wedge BD$$

(26) $$\frac{dr}{r} = dX \wedge \left(- \frac{AC}{XA \wedge AC} + \frac{AD}{XA \wedge AD} - \frac{BD}{XB \wedge BD} + \frac{BC}{XB \wedge BC} \right) +$$

$$+ dA \wedge \left(\frac{XC}{XA \wedge XC} - \frac{XD}{XA \wedge XD} \right) + dB \wedge \left(\frac{XD}{XB \wedge XD} - \frac{XC}{XB \wedge XC} \right) +$$

$$+ dC \wedge \left(- \frac{XA}{XA \wedge XC} + \frac{XB}{XB \wedge XC} \right) + dD \wedge \left(- \frac{XB}{XB \wedge XD} + \frac{XA}{XA \wedge XD} \right)$$

In the former expression (26) it is observed that the error depends mainly on the directions. Taking the unitary vectors and the sines of the corresponding angles, we have:

(27) $$\frac{dr}{r} = dX \wedge \left(\frac{1}{|XA|} \left(\frac{u_{AC}}{\sin \widehat{XAC}} - \frac{u_{AD}}{\sin \widehat{XAD}} \right) + \frac{1}{|XB|} \left(\frac{u_{BD}}{\sin \widehat{XBD}} - \frac{u_{BC}}{\sin \widehat{XBC}} \right) \right) +$$

$$+ \frac{dA}{|XA|} \wedge \left(\frac{u_{XC}}{\sin \widehat{AXC}} - \frac{u_{XD}}{\sin \widehat{AXD}} \right) + \frac{dB}{|XB|} \wedge \left(\frac{u_{XD}}{\sin \widehat{BXD}} - \frac{u_{XC}}{\sin \widehat{BXC}} \right) +$$

$$+ \frac{dC}{|XC|} \wedge \left(- \frac{u_{XA}}{\sin \widehat{AXC}} + \frac{u_{XB}}{\sin \widehat{BXC}} \right) + \frac{dD}{|XD|} \wedge \left(- \frac{u_{XB}}{\sin \widehat{BXD}} + \frac{u_{XA}}{\sin \widehat{AXD}} \right)$$

The error of X depends on the error of r (calculated from the photograph) and the errors of the position of A, B, C, D in the plane. Each of these errors is multiplied by a coefficient error that depends on the arrangement of the points in the plane. The error of X will be small when the vectors which are coefficients of dA, dB, dC, dD are small. Let us fix the modules of XA, XB, XC, XD and let us study how the angles influence the error of X. This error is minimal when all angles AXC, BXC, AXD and BXD are equal. This means that points A, B and X should be aligned in the plane as well as point C, D and X. The ideal arrangement of this four points to minimize the error is shown in figure 2.

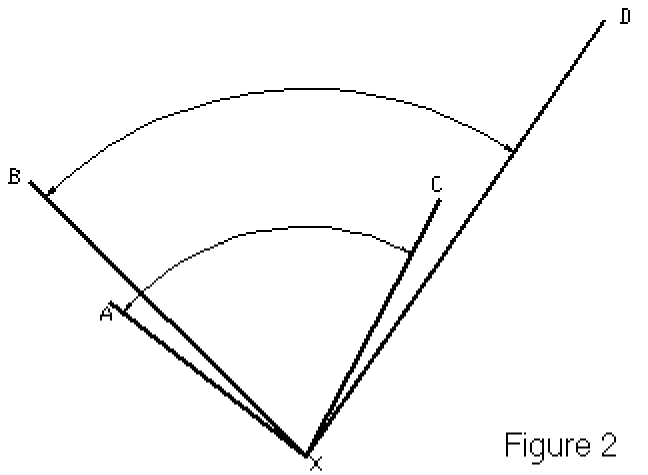

Figure 2

Reordering the previous equations (23-24) and writing them in a condensed form, we have:

(28) $$\frac{dr}{r} - dA \wedge v_A - dB \wedge v_B - dC \wedge v_C - dD \wedge v_D = dX \wedge v_X$$

(29) $$\frac{ds}{s} - dA \wedge w_A - dB \wedge w_B - dC \wedge w_C - dE \wedge w_E = dX \wedge w_X$$

where v_A, v_B, ... w_A, w_B, ... are the vectors being coefficient of dA, dB, ... in the first and second equalities, respectively.

The concept of the variance for a two-dimensional random variable is represented by a matrix. Let us suppose that X=(x, y) is a variable with the components partially correlated. In this case, in addition to the variances of x and y we have the covariance:

(30) $\mathrm{Var}(x) = E(x^2) - E(x)^2$ $\qquad$ $\mathrm{Var}(y) = E(y^2) - E(y)^2$

(31) $\mathrm{Cov}(x, y) = E(x\,y) - E(x)\,E(y)$

Let us consider now any variable z which is a linear combination of x and y:

(32) $z = \alpha x + \beta y$

Its variance is easily calculated:

(33) $\mathrm{Var}(z) = E([\alpha x + \beta y]^2) - E(\alpha x + \beta y)^2 = \alpha^2\,\mathrm{Var}(x) + \beta^2\,\mathrm{Var}(y) + 2\,\alpha\beta\,\mathrm{Cov}(x\,y)$

Written in matrix notation (Dunn, appendix 2, pp. 165-169):

(34) $$\mathrm{Var}(z) = (\alpha\ \beta)\begin{pmatrix} \mathrm{Var}(x) & \mathrm{Cov}(x, y) \\ \mathrm{Cov}(x, y) & \mathrm{Var}(y) \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

This equality shows that the variance is a symmetrical bilinear mapping which applied to a random variable z, understood as a vector of two components (α, β) in the base { x, y } gives the variance of z. When it is applied to two different variables, it gives their covariance. Let us consider another variable t, linear combination of x and y, and let us calculate the covariance with z:

(35) $t = \gamma x + \delta y$

(36) $\mathrm{Cov}(z, t) = E([\alpha x + \beta y][\gamma x + \delta y]) - E(\alpha x + \beta y)\,E(\gamma x + \delta y)$

$$= \alpha\gamma \mathrm{Var}(x) + (\alpha\delta + \beta\gamma)\,\mathrm{Cov}(x, y) + \beta\delta \mathrm{Var}(y)$$

(37) $$\mathrm{Cov}(z, t) = (\alpha\ \beta)\begin{pmatrix} \mathrm{Var}(x) & \mathrm{Cov}(x, y) \\ \mathrm{Cov}(x, y) & \mathrm{Var}(y) \end{pmatrix}\begin{pmatrix} \gamma \\ \delta \end{pmatrix}$$

The variance of the outer product of a random variable X = (x, y) multiplied by a constant vector v_X = (α, β) is:

(38) $$\mathrm{Var}(X \wedge v_X) = \mathrm{Var}(\beta x - \alpha y) = \beta^2 \mathrm{Var}(x) + \alpha^2 \mathrm{Var}(y) - 2\,\alpha\beta\,\mathrm{Cov}(x, y)$$

Written in matrix form:

(39) $$\mathrm{Var}(X \wedge v_X) = (\alpha\ \beta)\begin{pmatrix} \mathrm{Var}(y) & -\mathrm{Cov}(x, y) \\ -\mathrm{Cov}(x, y) & \mathrm{Var}(x) \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = v_X^t\, \mathrm{Var}^{adj}\, v_X$$

That is to say, the variance of the outer product is obtained applying the vectors v_X to the adjoint matrix of the variance. Using this equality, the variance matrix of X is determined. Taking into account that the cross ratios calculated from the photograph are independent variables of the coordinates measured in the plane, it is obtained:

(40) $$\frac{\mathrm{Var}(r)}{r^2} + v_A{}^t\, \mathrm{Var}^{adj}(A)\, v_A + v_B{}^t\, \mathrm{Var}^{adj}(B)\, v_B + v_C{}^t\, \mathrm{Var}^{adj}(C)\, v_C + v_D{}^t\, \mathrm{Var}^{adj}(D)\, v_D$$

$$= v_X{}^t\, \mathrm{Var}^{adj}(X)\, v_X$$

(41) $$\frac{\mathrm{Var}(s)}{s^2} + w_A{}^t\, \mathrm{Var}^{adj}(A)\, w_A + w_B{}^t\, \mathrm{Var}^{adj}(B)\, w_B + w_C{}^t\, \mathrm{Var}^{adj}(C)\, w_C + w_E{}^t\, \mathrm{Var}^{adj}(E)\, w_E$$

$$= w_X{}^t\, \mathrm{Var}^{adj}(X)\, w_X$$

(42) $$\frac{\mathrm{Cov}(r, s)}{r\,s} + v_A{}^t\, \mathrm{Var}^{adj}(A)\, w_A + v_B{}^t\, \mathrm{Var}^{adj}(B)\, w_B + v_C{}^t\, \mathrm{Var}^{adj}(C)\, w_C = v_X{}^t\, \mathrm{Var}^{adj}(X)\, w_X$$

Let us consider the particular case in which the variances of the coordinates of A, B, C, D are all equal and the horizontal coordinates are independent from the vertical ones, that is to say, the covariances are zero. This corresponds to considering the same error for any direction and therefore to supposing that the real position of a point stands within a circle centered at the measured position of this point. Var(P) will be the scalar variance of all these points. Then the equations become:

(43) $$\frac{\mathrm{Var}(r)}{r^2} + \mathrm{Var}(P)\,[\,v_A{}^2 + v_B{}^2 + v_C{}^2 + v_D{}^2\,] = v_X{}^t\, \mathrm{Var}^{adj}(X)\, v_X$$

(44) $$\frac{\mathrm{Var}(s)}{s^2} + \mathrm{Var}(P)\,[\,w_A{}^2 + w_B{}^2 + w_C{}^2 + w_E{}^2\,] = w_X{}^t\, \mathrm{Var}^{adj}(X)\, w_X$$

(45) $$\frac{\mathrm{Cov}(r, s)}{r\,s} + \mathrm{Var}(P)\,[\,v_A{}^t w_A + v_B{}^t w_B + v_C{}^t w_C\,] = v_X{}^t\, \mathrm{Var}^{adj}(X)\, w_x$$

Denoting by h, j, k the terms at the left hand of each equality, we have:

(46) $$h = v_X{}^t\, \mathrm{Var}^{adj}(X)\, v_X$$

(47) $$j = w_X{}^t\, \mathrm{Var}^{adj}(X)\, w_X$$

(48) $k = v_X^{\,t}\, Var^{adj}(X)\, w_X$

Writing with components $v_X = (a, b)$ i $w_X = (g, d)$, we build the matrix M:

(49) $$M = (v_X\ w_X) = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$$

Now the three previous equalities (46-49) are a matrix equality:

(50) $$\begin{pmatrix} h & k \\ k & j \end{pmatrix} = M^t\, Var^{adj}(X)\, M$$

Clearing the variance matrix one obtains:

(51) $$M^{-1,t} \begin{pmatrix} h & k \\ k & j \end{pmatrix} M^{-1} = Var^{adj}(X)$$

And from here, taking adjoint matrices:

(52) $$\frac{M}{\det M} \begin{pmatrix} j & -k \\ -k & h \end{pmatrix} \frac{M^t}{\det M} = Var (X)$$

Explicitly:

(53) $$Var (x) = \frac{h\,\gamma^2 + j\,\alpha^2 - 2\,k\,\alpha\,\gamma}{(\alpha\delta - \beta\gamma)^2}$$

(54) $$Var (y) = \frac{h\,\delta^2 + j\,\beta^2 - 2\,k\,\beta\,\delta}{(\alpha\delta - \beta\gamma)^2}$$

(55) $$Cov (x, y) = \frac{h\,\gamma\delta + j\,\alpha\beta - k\,(\alpha\delta + \beta\gamma)}{(\alpha\delta - \beta\gamma)^2}$$

The variances can become very large when the denominator is small:

(56) $$\alpha\delta - \beta\gamma = v_X \wedge w_X = v_X \wedge (w_X - v_X) = 0$$

We recall these vectors:

(57) $$v_X = \frac{1}{|XA|} \left(\frac{u_{AC}}{\sin \widehat{XAC}} - \frac{u_{AD}}{\sin \widehat{XAD}} \right) + \frac{1}{|XB|} \left(\frac{u_{BD}}{\sin \widehat{XBD}} - \frac{u_{BC}}{\sin \widehat{XBC}} \right)$$

(58) $$w_X = \frac{1}{|XA|} \left(\frac{u_{AC}}{\sin \widehat{XAC}} - \frac{u_{AE}}{\sin \widehat{XAE}} \right) + \frac{1}{|XB|} \left(\frac{u_{BE}}{\sin \widehat{XBE}} - \frac{u_{BC}}{\sin \widehat{XBC}} \right)$$

(59) $$w_X - v_X = \frac{1}{|XA|} \left(\frac{u_{AD}}{\sin \widehat{XAD}} - \frac{u_{AE}}{\sin \widehat{XAE}} \right) + \frac{1}{|XB|} \left(\frac{u_{BE}}{\sin \widehat{XBE}} - \frac{u_{BD}}{\sin \widehat{XBD}} \right)$$

When one of these vectors is null, the outer product of the denominator is zero. v_X vanishes when the angle XAC coincides with XAD and the angle XBC coincides with XBD. This happens when, whether

A, B, C, D are aligned or C and D are the same point. w_X vanishes when the angle XAC coincides with XAE and the angle XBE coincides with XBC. This happens either when points A, B, C and E are aligned or when C and E are the same point. Finally, $w_X - v_X$ vanishes when the angle XAD coincides with XAE and the angle XBE coincides with XBD. This occurs either when points A, B, D and E are aligned or when D and E are the same point. If D and E are close, the two conics collapse into one conic, yielding an indeterminate system.

The points A and B can not be close either, since in this case the terms of the first bracket are cancelled with those of the second bracket and the three vectors vanish. One reaches the same conclusion analyzing the angles in analogous expressions of these vectors as functions of the modulus of XC and XD:

$$(60) \qquad v_X = \frac{1}{|XC|}\left(-\frac{u_{AC}}{\sin XCA} + \frac{u_{BC}}{\sin XCB}\right) + \frac{1}{|XD|}\left(\frac{u_{AD}}{\sin XDA} - \frac{u_{BD}}{\sin XDB}\right)$$

Summarizing, to avoid large variances, one must prevent the AB direction from being the same as one of the pairs of points C, D and E. The alignment of these three points does not mean any increase in the error. On the other hand, it is also convenient that A and B are not close, as well as any pair of C, D and E (figure 3). The worse configuration, which one must avoid, is that the five points are aligned in the plane. In a preceding study of a panoramic photograph of the Aneto mountain and its range -in the Pyrenees- (Enríquez, photograph n. 71, pp 72-73), which we knew was taken at the Picada pass, we observed a great error for the location of this point. Later, we have seen that the five peaks of the chain chosen as references were aligned.

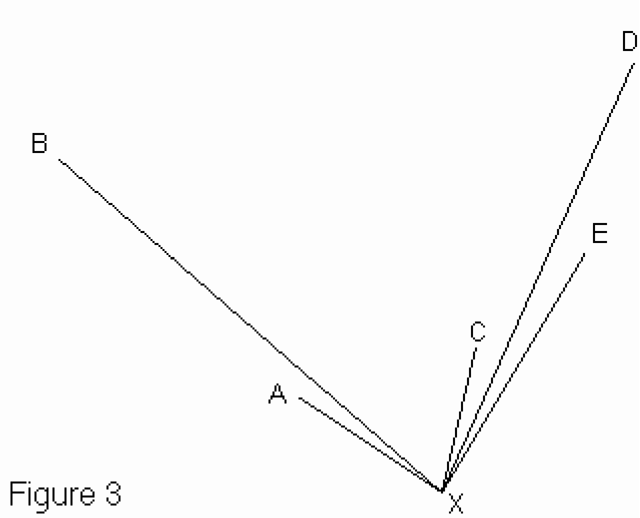

Figure 3

The error in the location of point X is not the same for all directions. Let x', y' be the coordinates of an axes system turned an angle θ with regard to system x, y:

$$(61) \qquad x' = x\cos\theta + y\sin\theta \qquad\qquad y' = -x\sin\theta + y\cos\theta$$

The error in the direction of x' and y' depends on the angle q, and it is given by the square root of its variance:

$$(62) \qquad Var(x') = \cos^2\theta\, Var(x) + 2\sin\theta\cos\theta\, Cov(x, y) + \sin^2\theta\, Var(y)$$

$$(63) \qquad Var(y') = \sin^2\theta\, Var(x) - 2\sin\theta\cos\theta\, Cov(x, y) + \cos^2\theta\, Var(y)$$

To find the minimal value of the variance, we equal to zero the derivative with respect to the angle.

$$\frac{d\,Var(x')}{d\theta} = 0$$

(64) $$\operatorname{tg} 2\theta = \frac{2\,\mathrm{Cov}(x, y)}{\mathrm{Var}(x) - \mathrm{Var}(y)}$$

from where we obtain two angles:

(65) $$\theta_1 = \frac{1}{2} \operatorname{arctg} \frac{2\,\mathrm{Cov}(x, y)}{\mathrm{Var}(x) - \mathrm{Var}(y)}$$

(66) $$\theta_2 = \frac{1}{2} \operatorname{arctg} \frac{2\,\mathrm{Cov}(x, y)}{\mathrm{Var}(x) - \mathrm{Var}(y)} + 90^\circ$$

These two angles determine the directions of minimum and maximum error (the principal error directions), which are perpendicular. We will indicate the coordinates taken in these directions by x_e and y_e, and they are statistically independent variables if we suppose that the errors have a normal distribution:

(67) $$x_e = x \cos\theta_1 + y \sin\theta_1 \qquad y_e = -x \sin\theta_1 + y \cos\theta_1 = x \cos\theta_2 + y \sin\theta_2$$

(68) $$\mathrm{Cov}(x_e, y_e) = (\cos\theta_1 \quad \sin\theta_1) \begin{pmatrix} \mathrm{Var}(x) & \mathrm{Cov}(x, y) \\ \mathrm{Cov}(x, y) & \mathrm{Var}(y) \end{pmatrix} \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix} = 0$$

Referred to the principal error coordinates, the variance matrix is diagonal. Any pair of coordinate axes will form an angle φ with the principal axes x_e ,y_e:

(69) $$\varphi = \theta - \theta_1 \qquad x' = x_e \cos\varphi + y_e \sin\varphi \qquad y' = -x_e \sin\varphi + y_e \cos\varphi$$

(70) $$\mathrm{Var}(x') = (\cos\varphi \quad \sin\varphi) \begin{pmatrix} \mathrm{Var}(x_e) & 0 \\ 0 & \mathrm{Var}(y_e) \end{pmatrix} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = \cos^2\varphi\,\mathrm{Var}(x_e) + \sin^2\varphi\,\mathrm{Var}(y_e)$$

The graph of the variance, or the standard deviation, is a curve resembling an erythrocyte seen from its side with both axes determined by the principal error directions. When the ratio $\mathrm{Var}(x_e) / \mathrm{Var}(y_e)$ is the unity, this figure becomes a circumference; when the ratio is large (or small) this figure becomes two adjacent quasi-circumferences. In fact, the inverse of the variance is an ellipse with semiaxes equal to the inverses of the variances of the principal error directions. In conclusion, the directions are not equivalent and it is observed that there is usually more error in a direction than in its perpendicular one. The trace of the variance matrix, which does not depend on the vectors base for which the bilinear mapping is expressed, can be taken as the global measure of the error.

(71) $$\mathrm{Var}(x') + \mathrm{Var}(y') = \mathrm{Var}(x) + \mathrm{Var}(y) =$$

$$= \frac{h(\gamma'^2 + \delta'^2) + j(\alpha'^2 + \beta'^2) - 2k(\alpha\gamma + \beta\delta)}{(\alpha\delta - \beta\gamma)^2} = \frac{h\,w_X'^2 + j\,v_X'^2 - 2k\,v_X \cdot w_X}{(v_X \wedge w_X)^2}$$

6 First example

Photograph 1: Photograph of the port of Barcelona taken from the Mirador de l'Alcalde. The points of reference are not optimum to minimize the errors, since they are distributed uniformly on the panoramic, instead of being concentrated around two points.
A: the memorial to Columbus.
B: the end of the ceiling of the France station railroad terminal.
C: the west corner of the Hotel Arts in the Vila Olímpica.
D: the tower Jaume I of the aerial tramway over the harbor.
E: the tower Sant Sebastià, also of the aerial tramway over the harbor.

The image has been digitized from the photograph on paper with a hand scanner and treated with the program *Finishing Touch*. The orthogonality of the projections has been adjusted using the sea horizon: it has been necessary to turn 1º the photograph. Enlarging the image, the horizontal positions of the five points can be found with much precision. The values obtained from this photograph, in the units of *Finishing Touch* are:

(72) A = 66 B = 216 C = 401 D = 757 E = 1044

We have located these points on a computerized map of Barcelona, the *VisualMap* (figure 4), obtaining the following coordinates:

(73) A = (-2691, 1631) B = (-1514, 1605) C = (-754, 1834) D = (-2844, 2058) E = (-2305, 2431)

Each unit of *VisualMap* corresponds to a meter of reality.

The points A, B, C and E remain clearly defined in the *VisualMap*. The position of point C, the west corner of Hotel Arts was not so clear and it was specified *in situ*. For the first coordinate, we have counted 140 steps from Francesc Aranda street to the Carles I Blvd., and the corner of the building was at the 49th step. To specify the second coordinate, we have counted 18 steps of the sidewalk.

The *VisualMap* has the coordinates origin at the center of the Plça. de les Glòries, the negative sense of x-axis points towards Plça. Espanya and the positive sense of the y-axis points towards the sea. In order to work with the customary orientation of the axis, we have changed the sign of the second coordinate for all the points and all the coordinates of points A, B, C, D and E with which we have made the calculations are then negative.

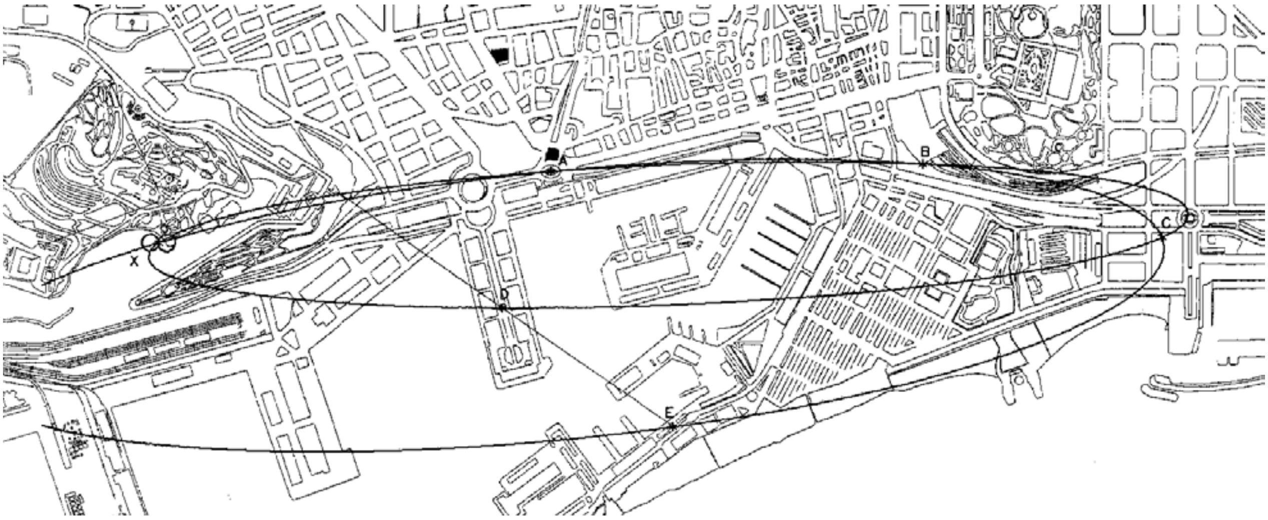

Figure 4

The value of the cross ratio r of the points A, B, C and D, calculated from the photograph is 1.41773. Using the program *Mathematica* with the instructions enclosed (annex 1), we have obtained the equation of the conic passing through these points with this cross ratio:

(74) $46670.5\, x^2 + 2.41136\cdot 10^6\, y^2 - 197063\, x\, y - 1.44143\cdot 10^8\, x + 8.36409\cdot 10^9\, y + 7.3663\cdot 10^{12} = 0$

In the same way, with the cross ratio s of the points A, B, C and E whose value is 1.53308, we have obtained the equation of the conic passing through these points:

(75) $113182\, x^2 + 1.98191\cdot 10^6\, y^2 - 371762\, x\, y - 1.58823\cdot 10^8\, x + 7.13671\cdot 10^9\, y + 6.75244\cdot 10^{12} = 0$

Both conics are ellipses. The intersection points of both conics are A, B, C and X. With *Mathematica* the coordinates of X have been calculated: (-3932.11, -1852.18). Rounding them with the signs of *VisualMap,* (-3932, 1852).

For the initial data, we evaluate in 2 units of *Finishing Touch* the error of the location of the points on the photograph, and in 3 units of *VisualMap* (3 meters) the error of the coordinates of the points on the map. With these data and using the enclosed program written with *Qbasic* (annex 2), the error calculation gives:

(76) Var(x) = 2688.792   x standard deviation = 51.85
Var(y) = 30.287   y standard deviation = 5.50
Cov(x, y) = - 101.467

(77) Maximum error direction : -2.18º with regard to the x axis with variance 2692.659
Minimum error direction : 87.82º with variance 26.420

The trace of the variance matrix is 2719.079 and, being independent of the direction, we have chosen it as a global measure of the error for this first example. The calculated coordinates of the point X with the standard deviations are:

(78) $(-3932 \pm 52,\ 1852 \pm 6)$

And for the *Visual Map*, the coordinates of the point of the Mirador de l'Alcalde from where the photograph was taken are:

(79) $(-3889 \pm 3,\ 1859 \pm 3)$

The conics and the error globule are plotted on the Barcelona map in figure 4.

7 Second example

Photograph 2: Photograph taken from the flat roof of the building 129 of Tarragona street looking onto the Eixample of Barcelona. On the photograph, we have chosen the points together in two sets in order to minimize the error and therefore there are almost two alignments on the map: XBA y XEDC. A and B were kept quite separated from each other and so were C, D and E.
A: the north corner of the Columbus building.
B: the line of separation between the buildings 10 and 12 of Diputació street.
C: the west corner of the Mapfre building at the Vila Olímpica.
D: the tower of the cathedral of Barcelona.
E: On the façade of the I.E.S. Joan Miró at Vilamarí street 78, the marked point in the photograph, easily identifiable.

The photograph was developed and recorded in a CD-ROM instead of a paper copy. The software to visualize the photograph in CD-ROM has allowed the conversion to TIF format, the usual format for the *Finishing Touch* program. The values so obtained in the units of this program are:

(80) $A = 1658 \quad B = 1611 \quad C = 355 \quad D = 251 \quad E = 205$

The coordinates of these points in *VisualMap* are:

(81) $A=(-2772, 1431) \quad B=(-4206, -126) \quad C=(-611, 1833) \quad D=(-2126, 888) \quad E=(-4108, -207)$

Only the C point is determined precisely on the *VisualMap*. In order to locate the other points, it has been needed to measure them *in situ*.

To work with the customary sense of the axes the sign of the second coordinate has been changed for all the points.

The cross ratio of points A, B, C and D is 1.00277. The conic passing through these points with this cross ratio is:

(82) $1.47413\cdot10^6\, x^2 + 2.16609\cdot10^6\, y^2 + 3.71588\cdot10^6\, x\, y + 1.12554\cdot10^{10}\, x + 1.44497\cdot10^{10}\, y + 2.13735\cdot10^{13} = 0$

The cross ratio of A, B, C and E is 1.00386. The conic passing through these points with this cross ratio is:

(83) $3.18868\cdot10^6\, x^2 + 5.03238\cdot10^6\, y^2 + 8.40305\cdot10^6\, x\, y + 2.501\cdot10^{10}\, x + 3.33782\cdot10^{10}\, y + 4.89507\cdot10^{13} = 0$

Both conics are hyperbolas. The coordinates of point X obtained with *Mathematica* are (-4463.48, 398.192); rounding with the signs of *VisualMap* are (-4463, - 398).

We evaluate in 3 units of *VisualMap* (3 meters) the error of the coordinates of points A, B, C, D and E, the same value of the first example, which allows us to compare the errors for both examples.

Evaluating the error of the situation of the points on the photograph is more complex. One unit of *Finishing Touch* on photograph 1 is not equivalent to a unit on photograph 2. In order to compare the errors in both examples one must find the equivalence between both units. The photograph 1 had initially a width of 15.2 cm. After digitizing with a scanner, turning and cutting out, it has a width of 14.75 cm. and 1153 units of *Finishing Touch*. Each of these units is equivalent to $1.28 \cdot 10^{-2}$ cm of the initial photograph on paper.

The photograph 2 was recorded in a CD-ROM. After cutting out, the final width is 61.18% of the initial width, which is equivalent to 1770 units of *Finishing Touch*. If a paper copy of the photograph 2 had been made with the same size as photograph 1, each unit would be equivalent to $15.2 \cdot 0.6118 / 1770 = 5.25 \cdot 10^{-3}$ cm of the paper copy. Comparing, each unit of photograph 1 is equivalent to 2.44 units of photograph 2.

Since in the first example we have evaluated in 2 units the error in the position of the points on the photograph, we must take 4.88 units for the error in photograph 2.

With these errors of the initial data (3 m in the map, 4.88 units in the photograph), the result of the error calculation is:

(84) $\text{Var}(x) = 186.247$ x standard deviation = 13.65
$\text{Var}(y) = 115.611$ y standard deviation = 10.75
$\text{Cov}(x, y) = -139.767$

(85) Maximum error direction: - 37.91º with variance 295.089
Minimum error direction: 52.09º with variance 6.769

The trace of the variance matrix is 301.858 which we take as the global measure of the error of this second example.

Since the errors in the initial data are equivalent in both examples, in the photograph as well as on the map, one may compare the error of the determination of point X using the trace of the variance matrix, which is independent of the directions. The error is much smaller in the second example than in the first because the trace of the variance matrix is 9 times smaller.

The calculated coordinates of point X with the standard deviation are:

(86) $(-4463 \pm 14, -398 \pm 11)$

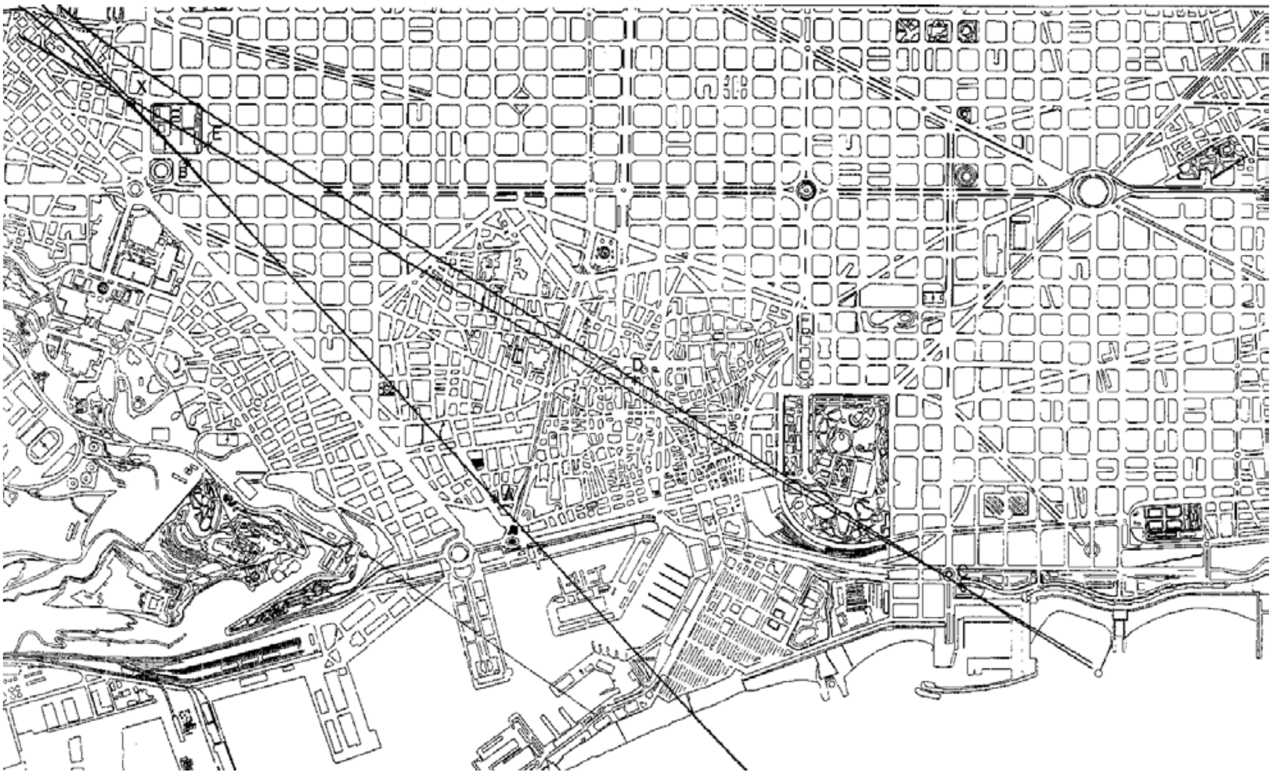

Figure 5

According to the *VisualMap*, the coordinates from where the photograph was actually taken are:

(87) (-4445 ± 3, -379 ± 3)

Without comparing both examples and measuring directly the error of the position of the points on the photograph 2, we value the error in 4 units because the photograph is not so bright. Casually, it differs little from 4.88 and the changes are minimal (trace of variance matrix 295.472) The error ranges of the calculated coordinates of X are the same as before.

The conics have been plotted on the *VisualMap* in the figure 5, together with the error globule, though it is too small to be appreciated. The scale of the map is different from that of figure 4.

8 Conclusions

The method of cross ratio is, after all, a generalization of the method of alignments. The method of alignments is the one people use by intuition when they attempt to find the position from where the photograph was taken. The method of alignments needs four points aligned in pairs with the observer. If they are not aligned, the cross ratio method must be used together with a fifth point. The error in the cross ratio method is minimum for the limiting case of the alignments method.

The calculations with *Mathematica* of the intersections have shown that the solution found for X does not depend on the pair of conics used, provided that the data of A, B, C, D and E are the same. The errors observed in the values of X do not come from calculation, but they have been propagated from the initial errors of the measurements.

The program of error calculation has allowed to prove that, the five reference points being fixed, the error of X does not depend on the assignment of labels A, B, C, D and E, that is, on the choice of the pair of conics among the ten possible conics.

These results answer Stewart's question (p. 98): "Shall there be a combination -such as the average, for example- of these ten solutions, whose error is the minimum possible?". The answer is no. The solution is always the same with the same error, which only depends on the error of the initial measurements. However, one can decrease the error using more solutions with a statistical sample obtained from choosing different sets of five points. This outlines the possibility of using two photographs or more, looking at any direction, with the only condition that they have been taken from the same place.

These results also contradict Stewart's affirmation (p.98): "...confirming thus our first result, but it has the advantage of cutting the first ellipse (and also, furthermore, the second one) in a greater angle, which facilitates a more precise determination of the intersection point P." The precision does not depend on the angles with which one conic cuts the other at point X.

There are always two principal directions of error, one of minimum error and the other of maximum error, which are orthogonal. The error in any other direction is a function of the angle between this direction and the principal error directions. With different examples studied aside from those herein shown, one observes that the maximum error direction is directed to the point distribution.

The error depends on the relative arrangement between the chosen points and in relation to the observation point, such as the examples 1 and 2 show. Since it depends on fifteen measurements (five for the photograph and ten for the map), the result has much error if these are not made very carefully.

9 Acknowledgments

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References

[1] DUNN, G. *Design and analysis of reliability studies. The statistical evaluation of measurement errors*. Oxford University Press, 1989.
[2] ENRÍQUEZ DE SALAMANCA, C. [author and editor], SICILIA, A.G.; *Panoramas del Pirineo Español*. Madrid, 1977.
[3] GRAÑÓ, N.; LLONA, A.; SUAU, F. "Geometria projectiva: El problema de la fotografia".*Butll. Soc. Cat. Mat.* 12-1 (1997) p. 73-83.
[4] GRASSMANN, H. *Teoría de la Extensión*. Spanish translation of *Die Ausdehnungslehre*, 1844, by Emilio Óscar Roxin. Espasa Calpe, Buenos Aires, 1947.
[5] MALACARA, D. [editor] *Geometrical and Instrumental Optics. Methods of Experimental Physics*, vol. 25. Academic Press, Inc., London, 1988.
[6] SIROHI, R. S.; KOTHIYAL, M. P. *Optical Components, Systems and Measurement Techniques*. Marcel Dekker, Inc. N. Y., 1991.
[7] STEWART, I., "Armoniosas relaciones (y razones no menos armónicas) entre el mapa y el territorio". *Investigación y Ciencia* (May 1990) p 93-98.
[8] STRAHLER, A. N. *Geografía Física*, 5th ed.,Omega. Barcelona, 1981.

Annex 1: file for the calculations of the example 1 with *Mathematica*

```
Rd[A_,B_,C_,D_]:=(C-A)(D-B)/(C-B)/(D-A); (*Definition of the cross ratio*)
(* Coordinates of the points on the map *)
P1={-2691.,-1631.};P2={-1514.,-1605.};P3={-754.,-1834.};P4={-2844.,-2058.};
P5={-2305.,-2431.};
(* Positions of the points on the photograph *)
P1'=66.; P2'=216.; P3'=401.; P4'=757.; P5'=1044.;
(* Conic passing through the points P1,P2,P3 and P4 with cross ratio r1 *)
r1=Rd[P1',P2',P3',P4']
f1[x_,y_]:=Expand[
(P1[[1]]P3[[2]]-P3[[1]]P1[[2]]+(P1[[2]]-P3[[2]])x+
(P3[[1]]-P1[[1]])y)(P2[[1]]P4[[2]]-P4[[1]]P2[[2]]+
(P2[[2]]-P4[[2]])x+(P4[[1]]-P2[[1]])y)-r1(
(P2[[1]]P3[[2]]-P3[[1]]P2[[2]]+(P2[[2]]-P3[[2]])x+ (P3[[1]]-P2[[1]])y)(P1[[1]]P4
[[2]]-P4[[1]]P1[[2]]+
(P1[[2]]-P4[[2]])x+(P4[[1]]-P1[[1]])y) )]
```

```
f1[x,y]
(* Conic passing through the points P1,P2,P3 i P5 with cross ratio r2 *)
r2=Rd<[>P1',P2',P3',P5']
f2[x_,y_]:=Expand[
(P1[[1]]P3[[2]]-P3[[1]]P1[[2]]+(P1[[2]]-P3[[2]])x+
(P3[[1]]-P1[[1]])y)(P2[[1]]P5[[2]]-P5[[1]]P2[[2]]+
(P2[[2]]-P5[[2]])x+(P5[[1]]-P2[[1]])y)-r2(
(P2[[1]]P3[[2]]-P3[[1]]P2[[2]]+(P2[[2]]-P3[[2]])x+
(P3[[1]]-P2[[1]])y)(P1[[1]]P5[[2]]-P5[[1]]P1[[2]]+
(P1[[2]]-P5[[2]])x+(P5[[1]]-P1[[1]])y) )]
f2[x,y]
(* Intersection of both conics *)
L=Solve[{f1[x,y]==0,f2[x,y]==0},{x,y}]
(* Plot of implicit functions *)
Needs["Graphics 'ImplicitPlot'"]
(* Plot of the conics *)
A=ImplicitPlot[{f1[x,y]==0,f2[x,y]==0},{x,-4000,-500},
{y,-3000,-1500}, (*adjusted to the example 1*) PlotPoints->100, Axes->None];
(* Plot of the points and the conics *)
B=ListPlot[{x,y}/.L, Axes->None];
Show[A,B,AspectRatio->Automatic];
```

Annex 2: Program for the error calculation written in QBasic

```
REM ************ error 19/6/95 **************
REM *********** calcul d'error en la fotografia **********
REM *** vrr=Var(r)/r^2 vss=Var(s)/s^2 cvrs=Covar(r,s)/(r*s) ********
INPUT "ef"; ef: vf = ef ^ 2
INPUT "a"; a: INPUT "b"; b: INPUT "c"; c: INPUT "d"; d: INPUT "e"; e
ab =b-a: ac =c-a: ad =d-a: ae =e-a: bc =c-b: bd =d-b: be =e-b: cd =d-c: ce =e-c
r = ac * bd / (ad * bc): s = ac * be / (ae * bc)
vrr =(cd/(ac*ad))^2 + (cd/(bc*bd))^2 + (ab/(ac*bc))^2 + (ab/(ad*bd))^2
vss =(ce/(ac*ae))^2 + (ce/(bc*be))^2 + (ab/(ac*bc))^2 + (ab/(ae*be))^2
cvrs = cd*ce/(ac^2*ad*ae) + cd*ce/(bc^2*bd*be) + ab^2/(ac*bc)^2
vrr = vrr * vf: vss = vss * vf: cvrs = cvrs * vf
REM ************** punts del pla **************
INPUT "ep"; ep: vp =ep^2: INPUT "a1"; a1: INPUT "a2"; a2: INPUT "b1"; b1: INPUT
"b2"; b2
INPUT "c1"; c1: INPUT "c2"; c2: INPUT "d1"; d1: INPUT "d2"; d2
INPUT "e1"; e1: INPUT "e2"; e2: INPUT "x1"; x1: INPUT "x2"; x2
REM ********** calcul de vectors ******************
ac1=c1-a1: ac2=c2-a2: ad1=d1-a1: ad2=d2-a2: ae1=e1-a1: ae2=e2-a2: be1=e1-b1: be2=e2-
b2
bc1=c1-b1: bc2=c2-b2: bd1=d1-b1: bd2=d2-b2: xa1=a1-x1: xa2=a2-x2: xb1=b1-x1: xb2=b2-
x2
xc1=c1-x1: xc2=c2-x2: xd1=d1-x1: xd2=d2-x2: xe1=e1-x1: xe2=e2-x2
REM ********* calcul de productes vectorials ******
xac=xa1*xc2 - xc1*xa2: xad=xa1*xd2 - xd1*xa2: xae=xa1*xe2 - xe1*xa2
xbc=xb1*xc2 - xc1*xb2: xbd=xb1*xd2 - xd1*xb2: xbe=xb1*xe2 - xe1*xb2
va1=xc1/xac - xd1/xad: va2=xc2/xac - xd2/xad
vb1=xd1/xbd - xc1/xbc: vb2=xd2/xbd - xc2/xbc
vc1=-xa1/xac + xb1/xbc: vc2=-xa2/xac + xb2/xbc
vd1=-xb1/xbd + xa1/xad: vd2=-xb2/xbd + xa2/xad
vx1=-ac1/xac + ad1/xad - bd1/xbd + bc1/xbc
```

```
vx2=-ac2/xac + ad2/xad - bd2/xbd + bc2/xbc
wa1=xc1/xac - xe1/xae: wa2=xc2/xac - xe2/xae
wb1=xe1/xbe - xc1/xbc: wb2=xe2/xbe - xc2/xbc
wc1=-xa1/xac + xb1/xbc: wc2=-xa2/xac + xb2/xbc
we1=-xb1/xbe + xa1/xae: we2=-xb2/xbe + xa2/xae
wx1=-ac1/xac + ae1/xae - be1/xbe + bc1/xbc
wx2=-ac2/xac + ae2/xae - be2/xbe + bc2/xbc
PRINT va1; TAB(20); va2; TAB(40); wa1; TAB(60); wa2
PRINT vb1; TAB(20); vb2; TAB(40); wb1; TAB(60); wb2
PRINT vc1; TAB(20); vc2; TAB(40); wc1; TAB(60); wc2
PRINT vd1; TAB(20); vd2; TAB(40); we1; TAB(60); we2
PRINT vx1; TAB(20); vx2; TAB(40); wx1; TAB(60); wx2
REM ******** calcul de variances ***********
REM vxx=Var(x) vyy=Var(y) cvxy=Covar(x,y)
h=vrr + vp*(va1^2 + va2^2 + vb1^2 + vb2^2 + vc1^2 + vc2^2 + vd1^2 + vd2^2)
j=vss + vp*(wa1^2 + wa2^2 + wb1^2 + wb2^2 + wc1^2 + wc2^2 + we1^2 + we2^2)
k=cvrs + vp*(va1*wa1 + va2*wa2 + vb1*wb1 + vb2*wb2 + vc1*wc1 + vc2*wc2)
det=vx1*wx2 - vx2*wx1
vxx = (h*wx1^2 + j*vx1^2 - 2*k*vx1*wx1)/det^2
vyy = (h*wx2^2 + j*vx2^2 - 2*k*vx2*wx2)/det^2
cvxy = (h*wx1*wx2 + j*vx1*vx2 - k*(vx1*wx2 + vx2*wx1)) / det^2
PRINT "h, j, k"; TAB(20); h; TAB(40); j; TAB(60); k
PRINT "matriu de var."; TAB(20); vxx; TAB(40); vyy; TAB(60); cvxy
REM ******* calcul direccions principals d'error ************
pi = 3.1415926535#
z1 = ATN(2 * cvxy / (vxx - vyy)) / 2
z2 = z1 + pi/2
vxe = vxx * COS(z1)^2 + 2 * cvxy * COS(z1) * SIN(z1) + vyy * SIN(z1)^2
vye = vxx * SIN(z1)^2 - 2 * cvxy * SIN(z1) * COS(z1) + vyy * COS(z1)^2
PRINT "angles(graus)"; TAB(20); z1 * 180 / pi; TAB(40); z2 * 180 / pi
PRINT ; "var. principals"; TAB(20); vxe; TAB(40); vye
```