Notes on the Kähler Calculus for the Summer School

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1 CHAPTER 1: Perspective on the Kähler Calculus for Experts on Clifford Analysis

1.1 Reconsidered Program of the Kähler part of the summer school

This is the first of a few chapters that I plan to post in this web site for potential participants in the Summer School, phases II and III. It provides an illustration for (mainly) Clifford mathematicians who would like to know what the Kähler calculus is about and what it offers. Starting with the second chapter, these notes will be used at the school for the actual teaching.

For the moment, the table of contents will not be removed, so that the change will be easier to appreciate. I have come to the conclusion that rather than teaching as much theory as needed for its multiple applications, I shall present some materials and produce immediately applications of it. Very occasionally I shall borrow a theorem (like a uniqueness theorem) to be proved in a future chapter. The chapters to be posted here will be only part of what will be lectured in Brasov, just enough to provide the flavor of it. Those who are in a position to understand these notes and plan to attend the conference might be able to start reading the pertinent Kähler papers on their own and ask, during the lectures, questions more profound that they might be asking otherwise.

As an example, the second chapter will deal with Kähler algebra of scalar-valued differential forms. It will be followed by a pair of applications: the theorem of residues and, unrelated to that, a new perspective on the issue of superposition, a concept to be replaced with decomposition, since it is through decomposition that the objects that one tries to superimpose are actually generated. Let me be more explicit. If the wave function is a member of the Kähler algebra, it can be written as a sum of pieces that belong to the ideals and also are solutions, but this is not sufficient to make them particle states. If you start with particle states (even mixing particles with antiparticles both with both options for handness) you do not get that intermediate state where you decompose but do not quite get particles, as one still needs something else for this. Of course, one can introduce concepts like matrix densities or something of the sort to deal with issues of this kind. But this is not as natural, or as rich or as reliable as starting with a solution of a wave equation that you decompose to get spinors which are not yet wave functions for leptons living in the same ideals.

The third chapter will deal with, among other topics, Kähler differentiation of scalar-valued differential forms, strict harmonic and harmonic differentials, Dirac type spinors (before we even consider a Dirac type Kähler equation, since their form has to do with theory of solutions with symmetry of exterior systems regardless of specifics), and so on. Applications: Real complex-like calculus in the plane, Helmholtz theorem for differential $1-$forms and $2-$forms in 3-D Euclidean space, and a hint at why there is room for theory on quantum collapse of the wave function and quantum teleportation in this calculus. These issues are intimately related to the issue of decomposition. One could say that these topics are aspects of the same theory, which grows in sophistication as we keep building more basic theory.

A fourth chapter will deal with the "Kähler-Dirac equation", or simply Kähler equation. A variety of applications for relativistic quantum mechanics follows: emergence of momentum operators, Pauli-Dirac equation for particles and antiparticles, post Pauli-Dirac Hamiltonian and an additional couple of topics.

At about this point, we shall have reached the third phase of the Summer School. It will deal with symmetry and conservation, a uniqueness theorem for differential forms, integration at different levels of generality for differentials whose exterior derivative and co-derivative are known, Lie and Killing operators, unified spin and orbital angular momentum, total argument momentum and its square, antiparticles, leptons and quarks.

1.2 Introduction

The Kähler calculus is a calculus of tensor-valued differential forms. So, we have a product of structures. The restriction of the Kähler calculus

(KC) to scalar-valued differential forms is the generalization of the exterior calculus that results from replacing the underlying exterior algebra with Clifford algebra. Clifford analysts are used to the replacement of exterior with Clifford. Then, what is special here?

In the KC, the interior derivative of a scalar-valued differential $1-$form is like the divergence, and the same derivative of a vector field is zero, not the divergence, as we shall see. That is such an obvious difference that one can understand it before learning the KC. This difference has its origin in the fact that there is a greater variety of elements in the full Kähler calculus than in other calculi. But the real important differences consist in what the KC can do and the Dirac calculus cannot. Of course, this chapter is an exception. It is meant to provide a perspective to those who can understand it. Everybody else should start with the next chapter. So, if upon reading this chapter you start not to understand, abandon it and go to the next one. Just before moving to the next chapter, please read the last four sections of the present one. I have given the exact places where the claims were proved by Kähler.

The files I may put in the web site of this summer school will be referred as chapters because they will be so, up to minor modifications in a book I am writing about what I shall call the Cartan-Kähler calculus. It was announced in my previous book "Differential Forms for Physicists and Mathematicians". I know some readers are looking forward to read it. There will be prior chapters on Clifford algebra and the exterior calculus. But, for the purposes of the summer school, this will be chapter one.

An outstanding feature of the KC is the ease with which one gets deep into the world of quantum mechanics, immediately acquiring new vistas since one enters if through a different door. So much so that we shall speak of the quantum mechanics, ab initio relativistic, that emerges as a virtual concomitant of the KC. At the end of this chapter, we shall briefly illustrate some deep results for quantum mechanics that Dirac's theory does not match. For the moment let us just say that the wave function in the "Kähler-Dirac" equation for scalar-valued differential forms is for members of the algebra in general, not only specifically for spinors (i.e. for members of ideals in this algebra).

Of course, there will be a lot of work to be done to revisit every piece of physics from the new perspective. Better yet, just a little bit of Kähler algebra will allow us, at the end of the next chapter, to start seeing how old problems can be seen in a new light. For the moment, I shall deal only with some computational issues to respond to the question, what is new?

1.3 Kähler and Cartan-Kähler calculi

As I mentioned in the second half of the biographical note of this web site, the KC was formulated in three papers. The translation of the title of the first of those papers is *Interior and exterior differential calculus*, and the translation of the third is *The interior calculus.* The author is dealing with the same calculus at the same level of generality. The only main difference is that the last of these two is far more comprehensive than the first one. So, it appears that he was not too sure at some time(s) about what title he should have given to his calculus. The issue we have with titles far transcends this one because of a far more important reason, as we are about to explain.

We are fully interested in Kähler's calculus for scalar-valued differential forms, not for tensor-valued differential forms, which we shall replace with Clifford-valued ones. In the last two of these three cases, we are dealing with tensors products of algebras. Then, sometimes, we shall consider a restriction to "mirror elements" (concept not needed at this point), which constitute a fourth structure. All four have to do with Clifford structure but only one is a Clifford algebra. Hence, to minimize clutter, I shall often use the term algebra not in the technical sense but in the more general sense of the dictionary, namely as a mathematical system that uses symbols and specially letters to generalize certain arithmetical operations and relationships. In this wide sense, all four of those structures qualify as algebras.

Eventually, we shall use the terms Kähler's algebra and KC to when we deal only with scalar-valued differential forms. When dealing with Clifford valuedness, we shall use the terms Cartan-Clifford algebra and calculus. It is worth noticing that Cartan considered curvatures as bivector valued differential 2-forms. We do not risk ignoring any applications with tensor-valued differential forms, since Kähler did not produce any application for them.

1.4 Kähler's differential forms

Kähler wrote his general differential forms as

$$u^{k_1 \dots k_\mu}_{i_1 \dots i_\lambda} = \frac{1}{p!} a^{k_1 \dots k_\mu}_{i_1 \dots i_\lambda \; l_1 \dots l_p} dx^{l_1} \wedge \dots \wedge dx^{l_p}, \tag{1}$$

where we are using Einstein's convention of summation over repeated indices. His excessive use of components is not a desirable feature for a calculus, but has a very useful consequence. It shows explicitly that we must consider two types of subscripts. They refer to two essentially different concepts: scalar-valued differential r-forms (quantities with just a l series of indices) and tensor-valued differential 0-forms or tensor fields (quantities with only k and or i series of indices). Hence, his calculi are ab initio different from all other known calculi that are based on Clifford algebra. But Kähler did not exhibit the basis elements that pertain to the i and k indices.

The affine curvature is a very well known example of the rare quantities that have indices of all three types. Consider some vector-valued differential 1-form. It need not be closed. For simplicity let us assume that it were what Cartan and Kaehler would call the exterior derivative of a vector field, though practitioners might refer to it with the name of covariant derivative, name which is here reserved for a different purpose. We have

$$d\mathbf{v} = dv^i \mathbf{e}_i + v^k d\mathbf{e}_k = (dv^i + v^k \omega_k^i)\mathbf{e}_i. \tag{2}$$

Differentiating next $d\mathbf{v}$, we obtain a vector-valued differential $2-$form, $dd\mathbf{v}$ (which happens to be zero in Euclidean space), but we shall ignore this (you may assume that the manifold is only approximately a Euclidean space). The components of $dd\mathbf{v}$ are not what in the tensor calculus one calls covariant derivatives. For that, we would have to differentiate $v^j_{;k}\phi^k \mathbf{e}_j$, where (ϕ^i) is the basis of covariant vector fields dual to the basis field $(\mathbf{e}_j)$, i.e. $\phi^i \llcorner \mathbf{e}_k = \delta^i_k$. Let us not overlook that, in the Kähler calculus, a differential $1-$form is not a covariant vector (i.e. a linear function of vectors) field, but a function of curves, evaluated by integration on a given curve.

We get five terms for $dd\mathbf{v}$, two from the differentiation of $dv^i \mathbf{e}_i$, and three more from the differentiation of $v^k \omega_k^i \mathbf{e}_i$. The first term, ddv^i is obviously zero. The second and third terms cancel each other out. We thus have

$$dd\mathbf{v} = v^k d(\omega_k^i \mathbf{e}_i) = v^k (d\omega_k^i - \omega_k^j \wedge \omega_j^i)\mathbf{e}_i. \tag{3}$$

Since $d\omega_i^k - \omega_i^j \wedge \omega_j^k$ is a differential $2-$form, it is of the form. Hence,

$$dd\mathbf{v} = v^i R^k_{i\,l_1 l_2} \omega^{l_1} \wedge \omega^{l_2} \mathbf{e}_k. \tag{4}$$

This says that the components of the vector-valued differential $2-$form $dd\mathbf{v}$ are $v^i R^k_{i\,l_1 l_2}$. We then define curvatures $\mho$ by

$$\mho \equiv R^k_{i\,l_1 l_2} \omega^{l_1} \wedge \omega^{l_2} \phi^i \mathbf{e}_k, \tag{5}$$

where all three types of indices are involved. Clearly $dd\mathbf{v}$ results from evaluating $\mho$ in $\mathbf{v}$,

$$\mho \llcorner \mathbf{v} = R^k_{i\,l_1 l_2} \omega^{l_1} \wedge \omega^{l_2} \phi^i \mathbf{e}_k \llcorner v^m \mathbf{e}_m = v^k (d\omega_k^i - \omega_k^j \wedge \omega_j^i)\mathbf{e}_i = dd\mathbf{v}. \tag{6}$$

For more on this approach to affine curvature and the use of bases ω of differential $1-$forms (which are of the essence further below), check in your library my book "Differential geometry for physicists and mathematicians". I do not know of any other book which stays close to this Cartanian way of doing modern differential geometry, or to the Kähler calculus for that matter.

1.5 Kähler's differentiation

In Kähler, all differentiations except Lie differentiation are based on his concept of covariant differential. He gives it ab initio as

$$\begin{aligned} d_h a^{k_1 \ldots k_\mu}_{i_1 \ldots i_\lambda \; l_1 \ldots l_p} = & \frac{\partial}{\partial x^h} a^{k_1 \ldots k_\mu}_{i_1 \ldots i_\lambda \; l_1 \ldots l_p} + \\ & + \Gamma^{k_1}_{hr} a^{r \ldots k_\mu}_{i_1 \ldots i_\lambda \; l_1 \ldots l_p} + \ldots + \Gamma^{k_\mu}_{hr} a^{k_1 \ldots r}_{i_1 \ldots i_\lambda \; l_1 \ldots l_p} \\ & - \Gamma^{r}_{hi_1} a^{k_1 \ldots k_\mu}_{r \ldots i_\lambda \; l_1 \ldots l_p} + \ldots + \Gamma^{r}_{hi_r} a^{k_1 \ldots k_\mu}_{i_1 \ldots r \; l_1 \ldots l_p} \\ & - \Gamma^{r}_{hl_1} a^{k_1 \ldots k_\mu}_{i_1 \ldots i_\lambda \; r \ldots l_p} + \ldots + \Gamma^{r}_{hl_r} a^{k_1 \ldots k_\mu}_{i_1 \ldots i_\lambda \; l_1 \ldots r_p}, \end{aligned} \tag{7}$$

where the gammas are the Christoffel symbols. Since this equation will probably put many readers off, let me start by saying that the author of these notes does not keep anything like this in his memory, nor does he look for this formula when needed. There is a better way of doing his differentiations. At this point, let me just make some helpful comments.

The differential of scalar-valued differential forms written in terms of Cartesian coordinates requires only the first of those four lines. That is good enough for many applications. If the differential form is scalar-valued but the coordinates are not Cartesian, d_h involves only the first and fourth lines, regardless of whether the metric is the Euclidean metric disguised by the use of an arbitrary coordinate system or whether it is a proper post-Euclidean Riemannian metric. Lines two and three are for the non-scalar valuedness but only when the affine connection is the Levi-Civita (LC) connection. If the connection were another one, those symbols would have to be replaced with the components of the given (metric compatible) affine connection of the space. But the fourth line would not change in this respect, as it does not depend on the connection but only on the metric and its derivatives through the Christoffel symbols. In addition to generalizing this formula for arbitrary connection, one could also replace the tensor-valuedness with Clifford valuedness. And let us not ignore either that vector-valuedness can take place in exterior algebra, Clifford algebra and tensor algebra contexts. It does not make a difference.

A more efficient, less cumbersome way to deal with "Kähler differentiation" resorts to inferring the specific form of the covariant derivative in each case from the equations of structure of the manifold. That is what we are about to do.

Let us give the basic ideas about Kähler differentiation, represented by the symbol ∂. We seek to find an operator ∂ such that $\partial \vee u$ ($= \partial \wedge u + \partial \cdot u$) will become

$$\partial u = du + \delta u, \tag{8}$$

with $\partial \wedge u = du$, and such that δu will be intimately connected with the divergence. In Cartesian coordinates, ∂ could be simply $dx^i \dfrac{\partial}{\partial x^i}$. But this is not good enough for general coordinates, as the form of the divergence in curvilinear coordinate attests to. For this reason, we shall rather seek ∂u in the form

$$\partial u = dx^i \vee d_i u. \tag{9}$$

for some covariant derivatives $d_i u$ canonically determined by the structure of the manifold and such that $dx^i \wedge d_i u = du$. The notation

$$\partial u = \partial \vee u = \partial \wedge u + \partial \cdot u$$

still is justified if we understand it to mean

$$\partial \wedge u = dx^i \wedge d_i u = du, \qquad \partial \cdot u = dx^i \cdot d_i u = \delta u.$$

We have just connected with the exterior calculus, which we are extending with this δu, once we define $d_i u$. This will be much richer than Ricci, or de Rham or Dirac theory since the context is much larger than in those theories. The proof will be in the pudding. We shall thus refer to δu as the interior derivative since it is a concept more comprehensive than divergence depending on what objects u the operator δ is applied to.

1.6 Kähler's differentiation through geometric structure

Assuming we had already computed du by the standard formula in the exterior calculus, we might try to infer $d_i u$ from $du = dx^i \wedge d_i u$. But du does not determine d_i or ∂, which is the reason why we have not displayed $dx^i \partial / \partial_i$. It would work for the exterior derivative. but it would not yield the right divergence. But there is one such solution that is canonically determined by the equations of the structure of the manifold endowed with a metric, regardless of whether the manifold has an affine structure or not.

Define a set of n differential forms ω^i's such that

$$ds^2 = \sum_{i=1}^{n} (\omega^i)^2, \tag{10}$$

for any specifically given quadratic symmetric differential form $ds^2 = g_{ij}dx^i dx^j$ $(i = i, ..., n)$. These ω^i are defined up to the most general rotations in dimension n.

As any good book on differential geometry shows, the system of equations

$$d\omega^i = \omega^j \wedge \omega^i_j, \qquad \omega_{ij} + \omega_{ji} = 0 \tag{11}$$

defines a set of ω^i_j's. For the same metric, we may consider the system

$$0 = d(dx^i) = \omega^j \wedge \alpha^i_j, \qquad d_h g_{kl} = 0,$$

the second of these equations being known as the statement that the covariant derivatives the metric is zero. The Christoffel symbols are defined by $\alpha^i_j = \Gamma^i_{jl} dx^l$.

The last two systems define the same mathematical object. This does not mean that the α^i_j and the ω^i_j are equal, or even the components of some object of grade two. The Γ^i_{jl} and the Γ'^i_{jl} defined by $\omega^i_j = \Gamma^i_{jl}\omega^l$ are related by non-tensorial equations known as the transformations of connections. The mathematical object of which the α^i_j's and the ω^i_j's are components is a differential form valued in the Lie algebra of the Euclidean group (actually of the affine extension of the Lie algebra of the Euclidean group; do not bother about these details) of dimension n. All those other differential forms are components. It is legal tender nevertheless to call them differential forms by making them so as a matter of definition. I shall try to explain this with what should amount to a simpler example.

The first element of orthonormal vector bases at some point of Euclidean space is not a vector, but a set of them: $\cos\phi\ \mathbf{i} - \sin\phi\ \mathbf{j}$. It is basis dependent. Once a basis has been chosen, say for $\phi = \pi/4$, we have a vector $(2)^{-1/2}\ \mathbf{i} - (2)^{-1/2}\ \mathbf{j}$. We can now take this vector and express it in terms of any other basis (in particular for $\phi = \pi/3$), where it is not its first element. This type of idea is involved at a far more sophisticated level in the difference between α^i_j and the ω^i_j. For a deep understanding of this, see my book "Differential geometry for physicists and mathematicians" if you do not know of any other book dealing with this subject (I do not any, except for information scattered over a variety of E. Cartan's papers). The book is now in 500 libraries, hopefully enough of them to have one in a library of your country, where institution could get it for you if it does not have it.

In order to do differentiation in terms of bases that are not coordinate bases, we first define $f_{/k}$ as given by $df = f_{/k}dx^k$. We then have

$$\begin{aligned} dv &= a_{i/j}\omega^j \wedge \omega^i + a_l \wedge d\omega^l = \omega^j \wedge a_{i/j}\omega^i + a_l\omega^i \wedge \omega_i{}^l = \\ &= \omega^j \wedge a_{i/j}\omega^i - a_l \Gamma_i{}^l{}_j \omega^j \wedge \omega^i = \omega^j \wedge (a_{i/j}\omega^i - a_l\Gamma_i{}^l{}_j\omega^i). \end{aligned} \tag{12}$$

By virtue of the relation of this formula to the structure of a manifold, it pertains to define Kähler's covariant derivative $d_j v$ of a differential $1-$form $v = a_i \omega^i$ as

$$d_j v = a_{i/j} \omega^i - a_l \Gamma_{i\ j}^{\ l} \omega^i \tag{13}$$

If ω^i is dx^i, the covariant derivative $d_j v$ becomes

$$d_j v = a_{i},_{j}\, dx^i - a_l \Gamma^l_{ij} dx^i. \tag{14}$$

For rectilinear coordinates, thus Cartesian in particular, d_j reduces to the partial derivative with respect to x^j since the Christoffel symbols then become zero. For the interior derivative, we then have

$$\delta v = dx^j \cdot d_j v = dx^j \cdot dx^i (a_{i},_{j} - a_l \Gamma^l_{ij}) = g^{ij} (a_{i},_{j} - a_l \Gamma^l_{ij}) = a^j,_{j} - a_l \Gamma^{jl}{}_{j}. \tag{15}$$

We have not written δv as $\partial \cdot v$ since d_j now is $\partial_j - \underline{\ \ }_l \Gamma^{jl}_j$. We use the underbar because a has different subscripts on the two terms of $d_j v$.

Let us return to $d\omega^i = \omega^j \wedge \alpha^i_j$. For differential $1-$forms dx^j, the exterior derivative, $d(dx^i)$, is zero, and so is $dx^j \wedge \alpha^i_j$. On the other hand, the connection equations are

$$d\mathbf{e}^i = -\omega^i_j \mathbf{e}^j, \tag{16}$$

from which we would infer $d_h \mathbf{e}^i$, and similarly for $d_h \mathbf{e}_i$. The computing of interior and Kähler derivatives will be easily achieved from these equations. It is really simple.

1.7 Perspective on two physical applications developed by Kähler himself

This preview for Clifford analyst would be only one half of an important picture without mentioning a couple of very important physical implications of the quantum mechanics that appears to spring out of it, without obvious alternative. Both of them are related to the fact that, unlike Dirac's theory, Kähler's quantum mechanics is not about spinors but about differential forms. Spinors, in the form of members of ideals of the Kähler algebra, are a concomitant of the treatment of solutions with symmetry of exterior systems, as any system of differential equations can be shown to be.

In his 1960 paper, Kähler starts with a profound treatment of partial differentiation of differential forms with respect to the angular coordinate associated with rotational symmetry. Eventually, both spin and angular momentum emerge from it, at par. Tracing spin from the end of Kähler's derivation to the beginning of his argument, we realize that spin starts its life in Kähler 's theory inside a partial derivative of a field conceived not as a probability amplitude, but as a more conventional type of field; not as a spinor or member of an ideal in the algebra, but as a member of the algebra itself independently of any ideals.

In addition to the elegance and reliability of Kähler's argument, this reading in reverse shows that a particle is "*some special part of the field*", not just something that mediates among the particles, since these are simply special configurations of it. This is consistent with Einstein's view, though not exactly, as expressed by his words in correspondence with Einstein: "To realize the essential point of atomistic theory, it is sufficient to have a *field* of high intensity *in a spatially small region* ...". This idea of Einsteinean pedigree finds natural implementation in the Kähler calculus.

As profound as this result is in connection with spin, there is another one as profound, namely the treatment of not charge, not antiparticles, not energy, but all three at the same time. Let me give an inkling of why this is so. A solution with time translation symmetry of a quantum equation is known to require a phase factor with time and energy in the exponent. But, if the equation is written as an exterior system (as in the Cartan-Kähler theory of systems of differential equations), one also needs as factors two idempotents that define complementary ideals. Each of these belong to the signs of charge for each value of the energy in the phase factor.

Why may we make such a statement about charge? Kähler already explained why. It is a consequence of the decomposition into two terms of the wave function of a Kähler-Dirac equation under time translation symmetry, one from each of two complementary ideals. The corresponding conservation law for the wave function is then composed of two parts, one for each of those two terms. The left hand side of this law becomes the sum of the left hand sides of two conservation laws, not the sum of two conservation laws. The small difference between the two pairs of similar terms emerging in the process led Kähler to identify a physical magnitude, the electromagnetic charge, which may have two signs. The same energy but two signs of something when the coupling is electromagnetic coupling led him to view the two terms as representing particle and antiparticle. There is no need for an infinite sea of negative energy solutions, the weirdness of this concept being kept virtually silent in modern quantum physics. If Kähler had done his work in the late 1920's, we would be asking in 2016: "Dirac who?" Of course, Dirac was a genius since he provided a solution, though imperfect, to a problem for which the mathematics was not yet ripe.