CHAPTER 2: Kähler Algebra

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Participants in this summer school will at this point know enough Clifford algebra not to be confused if we sometimes represent Clifford products with the inverted wedge product sign and at other times by juxtaposition.

The Kähler calculus is a little bit strange. This may be because it is an approximation to something much deeper and to which I shall refer as the Cartan-Kähler calculus and which reconciles the ways of Cartan in differential geometry with the ways of Kähler on matters of calculus. One gets the impression that Kähler did not totally absorb Cartan's teachings in differential geometry. For this reason we shall use the term Kähler geometry and calculus when dealing only with scalar-valued differential forms. This less comprehensive version is enough to produce a great simplification of proofs and mathematical arguments when compare with the same issues in the standard mathematical and physical paradigms.

1 A practical approach to Kähler algebra

Hervibores in the African savanna are born knowing how to walk and soon learn how to run. They must have acquired some concept of running by the time one of them spots a lion and sends a signal to the herd and everybody starts to run. Of course, they do not know the anatomy and physiology involved in running but they surely know how to run. I do not need to explain how this example applies to this section. Thus, after reading section 1, one can jump to section 3 of this chapter and proceed with the integrations that are usual in an undergraduate course in complex variable. But, for the developments in the next chapters, one will have to read sections 1, 2 and 4. Readers who may not like the Kähler calculus are invited to develop alternatives that will efficiently match Kähler's results.

1.1 Definition of basic products

Kähler algebra (of scalar-valued differential forms) on a differentiable manifold (not on a tangent space or cotangent space!!!) endowed with a metric is the Clifford algebra defined by the relation

$$dx^i dx^j + dx^j dx^i = 2g^{ij}. \tag{1}$$

Hence

$$dx^i \wedge dx^j + dx^j \wedge dx^i = 0, \qquad dx^i \cdot dx^j = g^{ij} = \frac{1}{2}(dx^i dx^j + dx^j dx^i), \tag{2}$$

which come together as

$$dx^i dx^j = dx^i \wedge dx^j + dx^i \cdot dx^j. \tag{3}$$

If the differentiable manifold is a Euclidean vector space and the coordinates are Cartesian, we simply replace g^{ij} with δ^{ij}. All this can be said in a very elegant manner without using bases, but the definition would be very abstract. We have said emphasized the "of scalar-valued differential forms" to make sure we avoid confusions. Kähler also considered a more general structure consisting of tensor-valued differential forms. These do not constitute a Clifford algebra but the tensor product of a tensor algebra by the Kähler algebra just defined.

Let (A, B) be an ordered pair of two points in a differentiable manifold. Let γ be any curve with ends at those points. By definition dx^i is the function of curves such that

$$\int_\gamma dx^i = x^i_B - x^i_A. \tag{4}$$

Notice that we have not invoked either linear functions (i.e. covariant vectors) or covariant vector fields.

Consider next the differential form $3dx + xdy$ on curves between those same points. The integration depends on curves since this differential form does not have a potential function. Finally compute (To avoid confusion, I did not say evaluate) $3dx + xdy$ at any point with coordinate x equal to 2. We obtain $3dx + 2dy$. This is not something that we can evaluate on γ because it is not legal to first compute it at a point and then evaluate it on a curve, unless the differential form to be evaluated had been defined as $3dx + 2dy$ in the first place. The most we can do in this respect is to associate with $3dx + xdy$ equal to 2 points the linear function of tangent vectors $3\phi^1 + 2\phi^2$ where $\phi^i \llcorner \mathbf{a}_i = 1$, $\phi^i \llcorner \mathbf{a}_j = 0$ for $i \neq j$.

The pair (A, B) is the boundary of the manifold γ. So, Eq. (4) is a particular case of Stokes generalized theorem, which is here used as a definition, the reversion of roles of theorems and definitions being permissible. The formula for the exterior derivative would be a theorem rather than a definition. This may not be the most expedient course of action for the development of a calculus, but it is the most fundamental since integration requires less restrictive conditions than differentiation. Notice that we have not given any rule about differentiation yet, since none was needed.

One would proceed in similar manner for differential $r-$forms, i.e. as functions of $r-$surfaces. The use of Stokes theorem as a definition is nothing new. Elie Cartan already used it almost a century ago for defining the exterior derivative of a differential form whose coefficients are not differentiable functions.

Although we should not care too much at this point about the significance of the dot product of two differential $1-$forms, let us make a few remarks that may be helpful and a preview of arguments to come. In Cartesian coordinates, products $dx^i \cdot dx^j$ are zero unless $i = j$. But $dx^i \cdot dx^i$ does not have an invariant meaning; $\sum_1^n dx^i \cdot dx^i$ does. One would then have to see these products in context. $\sum_1^n dx^i \cdot dx^i$. This expression is related to some beautiful canonical Kaluza-Klein geometry which supersedes standard differential geometry. Again, we shall have to wait for Cartan-Kähler calculus in order to go deeper into it in a future chapter. Without going into any of that and not even raising the issue, Kähler obtained great results with the structure just mentioned. They eliminate significant problems of standard quantum mechanics.

Kähler used bases of differentials of arbitrary systems of coordinates, Cartesian coordinates not existing in non-Euclidean or non-pseudo-Euclidean spaces. Hence, we would have

$$dx^i dx^j + dx^j dx^i = 2\delta^{ij}. \tag{5}$$

in the Cartesian case. We would have to also replace g^{ij} with δ^{ij} in (2).

We prefer to use bases (ω^l) constituted by linear combinations of the differentials of the coordinates, whether these are Cartesian or not. We shall again have

$$\omega^i \omega^j + \omega^j \omega^i = 2g^{ij} \tag{6}$$

But we can always orthonormalize the metric so that we get

$$\omega^i \omega^j + \omega^j \omega^i = 2\delta^{ij}, \tag{7}$$

regardless of whether the manifold is a Euclidean space or not.

The ω^i notation is common in differential geometry developed with differential forms, usually in context of theory of connections. But this notation has nothing to do with connections, just with the metric. Of course, one can get Christoffel symbols and the Levi-Civita connection from the metric and its derivatives, but they are not needed at all for present purposes.

Let ω_i be defined by $\omega^j \cdot \omega_i = \delta^j_i$ for all pairs of indices. The ω_i's are not meant to be linear functions on the module spanned by the basis (ω^j) (The concept of module is more general than that of vector space, but you will not have problems here if you think of a module as if it were a vector space). Nothing could be more misleading than to think of the ω_i's as linear functions of anything. They are specific cases of functions of curves, just as legitimate as the ω^i's. They constitute just alternative bases in the module of differential forms.

Kähler likes to define the symbol e_i where we use the left dot multiplication "$\omega_i \cdot$". The action of e_i has the distributive property since, as we know,

$$\omega^i \cdot (\omega^j \wedge \omega^l \wedge \omega^k) = (\omega^i \cdot \omega^j)(\omega^l \wedge \omega^k) - (\omega^i \cdot \omega^l)(\omega^j \wedge \omega^k) + (\omega^i \cdot \omega^k)(\omega^i \wedge \omega^j. \quad (8)$$

And similarly for "$\omega_i \cdot (\omega^j \wedge \omega^l \wedge \omega^k)$", since ω_i is as legitimate a differential $1-$form as ω^i. We thus have

$$\omega_i \cdot (\omega^j \wedge \omega^l \wedge \omega^k) = (\omega_i \cdot \omega^j)(\omega^l \wedge \omega^k) - (\omega_i \cdot \omega^l)(\omega^j \wedge \omega^k) + (\omega_i \cdot \omega^k)(\omega^i \wedge \omega^j \quad (9)$$

or, equivalently,

$$e_i(\omega^j \wedge \omega^l \wedge \omega^k) = (e_i \omega^j)(\omega^l \wedge \omega^k) - (e_i \omega^l)(\omega^j \wedge \omega^k) + (e_i \omega^k)(\omega^i \wedge \omega^j. \quad (10)$$

Some learned readers would prefer to base the calculus and even differential geometry on vector field equations that largely parallel those we have given. A participant in this summer school already did so. The advantage of the Kähler way is that he deserves vector fields for other purposes. The parallelism ceases as soon as we consider differentiation. In any case and for the moment, I am reproducing what Kähler did and shall later show my view of where one should go, guided by the applications that ensue from the course of action to be followed. At this point I advocate the Kähler course of action because of the results he obtained and which nobody else appears to have even matched.

1.2 Using D=2 to get used to Kähler algebra

In terms of polar coordinates in 2-D Euclidean space E_2, we have

$$((d\rho)^2 = 1, \qquad (d\phi)^2 = d\phi \cdot d\phi = \frac{1}{\rho^2}, \qquad d\rho \cdot d\phi = 0. \quad (11)$$

$$dxdy = -dydx, \qquad (dxdy)^2 = -1. \tag{12}$$

We use the abbreviation i for $dxdy$. The complex-like inhomogeneous differential form z,

$$z \doteq x + ydxdy \doteq x + yi, \tag{13}$$

emerges from the relation between $(d\rho, d\phi)$ and (dx, dy):

$$d\phi = \frac{xdy - ydx}{x^2 + y^2} = \frac{x - ydxdy}{x^2 + y^2}dy = \frac{1}{x + ydxdy}dy = z^{-1}dy, \tag{14}$$

$$d\rho = \frac{xdx + ydy}{(x^2 + y^2)^{1/2}} = \rho\frac{x - ydxdy}{x^2 + y^2}dx = \frac{\rho}{x + ydxdy}dx = \rho z^{-1}dx. \tag{15}$$

By virtue of (13), it is clear that

$$z^{\pm m} = (x + yi)^{\pm m} = \rho^{\pm m}e^{m\phi i} = \rho^{\pm m}(\cos m\phi \pm i \sin m\phi), \tag{16}$$

for integer m.

There is a laudable effort on the part of Clifford mathematicians to replace the imaginary unit with elements of real Clifford algebra, an effort with which this author wholly agrees. But I see that there has been an abuse of the replacement of the imaginary unit with the unit pseudo scalar of the algebra when some other element in the algebra of square minus one is a more natural choice. So, rather than replace, it is a better process to let the right element emerge without resort to a replacement, which is what we have done in this case and shall be doing time and time again. And nothing impedes to proceed in reverse, and abbreviate (once found) $dxdy$ as i, and write $\cos m\phi \pm dxdy \sin m\phi$ as $e^{im\phi}$. e shall sometimes use $dxdy$ and i simultaneously, choosing one or the other in each specific case depending on what we wish to emphasize Finally, here is a remark for those without much Clifford experience who may have read this paragraph. The square of the unit pseudo-scalar may be plus or minus one. It depends on dimension and signature.

Let α be a differential $1-$form and let u and v be scalar functions. We have

$$(u + vi)\alpha = \alpha(u + vi)^*, \tag{17}$$

where

$$(u + vi)^* \equiv u - vi, \tag{18}$$

and, in particular,

$$z^* = x - yi, \quad z^* = \rho^2 z^{-1} \quad (z^*)^{-1} = \rho^{-2}z, \tag{19}$$

with $i = dxdy$. Clearly

$$u = \frac{(u + vi) + (u + vi)^*}{2}, \qquad v = \frac{(u + vi) - (u + vi)^*}{2i}, \tag{20}$$

as in the calculus of complex variable but again with $i = dxdy$.

1.3 The angular integrand

For the purpose of certain integrations,we wish to have the angular integrand part $j(\rho,\phi)d\phi$ of a differential 1−form $\alpha = h(\rho,\phi)d\rho + j(\rho,\phi)d\phi$, when it is given in terms of Cartesian coordinates

$$\alpha = k(x,y)dx + g(x,y)dy. \tag{21}$$

We clearly have

$$j = \rho^2 \ \alpha \cdot d\phi \tag{22}$$

and, therefore,

$$\alpha = wdx, \qquad w \equiv k - gdxdy = k - gidy. \tag{23}$$

We proceed to compute j. For that purpose, we express the dot product in terms of Clifford products:

$$j = \rho^2(wdx)\cdot(z^{-1}dy) = \frac{\rho^2}{2}\left[wdxz^{-1}dy + z^{-1}dywdx\right] =$$

$$= \frac{\rho^2}{2}\left[w(z^*)^{-1}i - z^{-1}w^*i\right] = \frac{1}{2}\left[wz - w^*z^*\right]i = -(wz)^{(2)}, \tag{24}$$

where the superscript refers to the coefficient of the differential 2−form part (of wz in this case). For the last step, we have used the last of (19).

2 Algebraic background for (algebraic) neophytes

A legitimate question is. Since Kähler algebra is just one more Clifford algebra, why should one give a name to it? One does not give a name to every possible Clifford algebra. The point is that the concept of differential form in 99% of the literature (though not in Rudin's classic book "Principles of Mathematical Analysis") is not as an integrand but as antisymmetric multilinear functions of vector fields. The appellative Kähler algebra is meant to remind us of this feature, of the need to remember that these are r−integrands, i.e. functions of r−surfaces.

2.1 Splits and pseudo-splits of a Clifford algebra

As you already know, the elements of even grade of a Clifford algebra constitute an algebra by themselves, called the even subalgebra. The set of the odd elements is not an algebra, since it is not closed (The product of two odd element is even). But there are other subalgebras of the same dimension, 2^{n-1}. Choose just an element of a basis of differential 1−forms. Call it dx^i for any given i. Any differential form u in the Kähler algebra can be written as

$$u = u' + dx^i \wedge u'', \tag{25}$$

where both u' and u'' are both uniquely defined if we demand that none of them contains dx^i as a factor. It is easy to prove that this decomposition splits the algebra into a subalgebra of dx^i's and the set of all the other elements in the algebra. If the signature of the algebra is definite, the signature of the subalgebra does not depend on which dx^i we choose. All of them are isomorphic. But, if it is not definite, we get different subalgebras depending on whether $(dx^i)^2$ is 1 or -1. For simplicity, we took a member dx^i of a basis of differential 1-forms. But any differential $1-$form can be chosen as a member of any such basis. Vice versa, we could always express the dx^i in the displayed formula as a linear combination of the differential of some other coordinate system.

In the following, we proceed very slowly, since we are in a hurry. Consider the "identity"

$$1 = \frac{1}{2}(1+a) + \frac{1}{2}(1-a), \tag{26}$$

where a is a member other than a scalar of Kähler algebra. Premultiplying by arbitrary elements u of the algebra, we get

$$u = u\frac{1}{2}(1+a) + u\frac{1}{2}(1-a) \tag{27}$$

This looks like a decomposition. Call it that if you wish, but let us play with it in order to distinguish between two situations.

Assume we had a two dimensional space with signature (1,1). To be specific, $(dx)^2 = 1$ and $(dt)^2 = -1$. In connection with the decomposition

$$1 = \frac{1}{2}(1+dx) + \frac{1}{2}(1-dx), \tag{28}$$

let us premultiply $\frac{1}{2}(1 \pm dx)$ by dx. We obtain

$$dx\frac{1}{2}(1 \pm dx) = \pm\frac{1}{2}(1 \pm dx). \tag{29}$$

The right and left of this equation are of the same type in the sense that they both have the factor $(1+dx)$ on the right. We shall later see that they cannot be written with the factor $(1-dx)$ as last factor on the right.

Consider on the other hand

$$1 = \frac{1}{2}(1+dt) + \frac{1}{2}(1-dt), \tag{30}$$

and premultiply $\frac{1}{2}(1+dt)$ by dt. We obtain

$$dt\frac{1}{2}(1+dt) = -\frac{1}{2}(1-dt). \tag{31}$$

This element can be written in both ways, meaning that in one case the last factor on the left is of the type $(1+a)$ and the last factor on the right is of the type $(1-a)$ for the same a.

Let us see another example. Simple operations show that whereas

$$\frac{1}{2}(1+dx^i)\frac{1}{2}(1+dx^i) = \frac{1}{2}(1+dx^i), \tag{32}$$

we, on the other hand, have

$$\frac{1}{2}(1+dt)\frac{1}{2}(1+dt) = \frac{1}{4}(1+dt) - \frac{1}{4}(1-dt). \tag{33}$$

Whereas on the right hand side of (32) the factor $(1+dx^i)$ remains, we have both $(1+dt)$ and $-\frac{1}{4}(1-dt)$ in the second. It makes a great difference whether a^2 equals 1 or -1. The cases $a^2 = +1$ has advantages that no alternatives have.

We shall refer to the equation

$$u = u\frac{1}{2}(1+dx^i) + u\frac{1}{2}(1-dx^i), \tag{34}$$

as a split of u, and, since u is arbitrary, it splits the algebra into two subalgebras without unit. On the other hand, the equation

$$u = u\frac{1}{2}(1+dt) + u\frac{1}{2}(1-dt) \tag{35}$$

does not represent a split since one can write any member of the algebra with both $(1+dt)$ and $(1-dt)$ as last factor. All this will become increasingly obvious as we familiarize ourselves with this type of computation in the next subsection.

Assume now that $\frac{1}{2}(1 \pm dx^i)$ were associated with space translation symmetry in the x^i direction. We would then expect that $\frac{1}{2}(1 \pm dt)$ would be associated with time translation symmetry. The sign of the square will make great difference vis a vis the decompositions that we are about to consider. Since one needs square $+1$ for interesting and fruitful results, Kähler used the decomposition

$$u = u\frac{1}{2}(1+idt) + u\frac{1}{2}(1-idt) \tag{36}$$

to treat time translation symmetry, with i as the usual imaginary unit of the calculus of complex variable. Notice that, since $(idt)^2$ equals $+1$, the last equation represents a split into two subalgebras. In order to achieve this behavior (which is desired because of what we shall see in the next section), Kähler resorted to the field of the complex numbers, if we were in the field of the reals in the first place. We shall see in a future chapter that the usual i can and should be represented by real elements of some structure. These elements will emerge spontaneously rather than be introduced ad hoc.

2.2 Idempotents and ideals

A subset A of some Clifford algebra Cl is said to be a left ideal if and only if

$$Cl\ A = A. \tag{37}$$

In words, A is a subset of Cl such that multiplying it on the left by any element of Cl returns an element of A. We are thus saying that A is closed under multiplication by Cl on the left.

An idempotent is defined as an element of Cl whose square is equal to itself. By trivial recursion, one sees that any integer power of an idempotent returns it. Some examples of idempotents are $\frac{1}{2}(1 \pm dx^i)$, $\frac{1}{2}(1 \pm idt)$ and $\frac{1}{2}(1 \pm idxdy)$. The two idempotents in each pair annul each other, i.e.

$$\frac{1}{2}(1+a)\frac{1}{2}(1-a) = \frac{1}{2}(1-a)\frac{1}{2}(1+a) = 0. \tag{38}$$

Any such pair naturally defines a complementary pair of left ideals

$$Cl = Cl\frac{1}{2}(1+a) + Cl\frac{1}{2}(1-a) \tag{39}$$

These ideals are subalgebras, but without a unit. let us start by showing that no element of the algebra except zero can be in both ideals at the same time. Indeed imagine you had

$$u\frac{1}{2}(1+a) = v\frac{1}{2}(1-a). \tag{40}$$

If we right multiply by $1-a$, we find that v is zero; and, if by $1+a$, we find that u is zero. The unit is a linear combination of $(1+a)$ from one ideal and $(1-a)$ from the other.

We now get some important practice with idempotents. Because of our future use of them, we shall adopt the same terminology as Kähler. Define idempotents

$$\epsilon^{\pm} \equiv \frac{1}{2}(1 \mp idt), \qquad \tau^{\pm} \equiv \frac{1}{2}(1 \pm idxdy). \tag{41}$$

Notice, but do not worry, about the inversion of sign between the left and right hand sides of the definition of the $\epsilon^{\pm}$. We have

$$\epsilon^+\epsilon^- = \epsilon^-\epsilon^+ = 0, \qquad \tau^+\tau^- = \tau^-\tau^+ = 0 \tag{42}$$

and

$$\epsilon^+ + \epsilon^- = 1, \qquad \tau^+ + \tau^- = 1. \tag{43}$$

We shall refer with the term of complementary idempotents to any pair of them that add up to one and mutually annul. Notice also the most important feature that the ϵ's commute with the τ's.

A remark about notation. Whereas $\epsilon^{\pm}\tau^{\pm}$ means the two options $\epsilon^{+}\tau^{+}$ and $\epsilon^{-}\tau^{-}$, we shall use the asterisk, as in $\epsilon^{\pm}\tau^{*}$, to mean the four options $\epsilon^{\pm}\tau^{\pm}$ and $\epsilon^{\pm}\tau^{\mp}$. The four $\epsilon^{\pm}\tau^{*}$ mutually annul. This is to be compared with the idempotents jointly generated by the non commuting elements idt and dx^{l} for given l. The four idempotents $\epsilon^{\pm}\frac{1}{2}(1{*}dx^{i})$ do not mutually annul. For example, we have ,

$$\left[\frac{1}{2}(1+idt)\frac{1}{2}(1+dx^{i})\right]\left[\frac{1}{2}(1+idt)\frac{1}{2}(1-dx^{i})\right]=\frac{1}{8}(1+idt)(1-dx^{\cdot i}). \tag{44}$$

The right hand side of

$$1=\epsilon^{+}\tau^{+}+\epsilon^{+}\tau^{-}+\epsilon^{-}\tau^{+}+\epsilon^{-}\tau^{-} \tag{45}$$

is a sum of mutually annulling idempotents, but the right hand side of

$$1=\epsilon^{+}\frac{1}{2}(1+dx^{i})+\epsilon^{+}\frac{1}{2}(1-dx^{i})+\epsilon^{-}\frac{1}{2}(1+dx^{i})+\epsilon^{-}\frac{1}{2}(1-dx^{i}) \tag{46}$$

is not.

We have extended the splits

$$Cl=Cl\ \epsilon^{+}+Cl\ \epsilon^{-},\qquad Cl=Cl\ \tau^{+}+Cl\ \tau^{-} \tag{47}$$

into the more comprehensive split

$$Cl=Cl\ \epsilon^{+}\tau^{+}+Cl\ \epsilon^{+}\tau^{-}+Cl\ \epsilon^{-}\tau^{+}+Cl\ \epsilon^{-}\tau^{-}. \tag{48}$$

It is legitimate to ask whether we can continue this extension. For that, we cannot count on $\frac{1}{2}(1\pm dx^{i})$ because of what we said above. We may wonder, however, whether we could find some idempotents other than the $\frac{1}{2}(1\pm dx^{i})$ in order to make the split even more comprehensive. For instance, $dtdxdy$ is of square $+1$. But we then have, for example,

$$\epsilon^{-}\tau^{+}\frac{1}{2}(1+dtdxdy)=0,\qquad \epsilon^{-}\tau^{+}\frac{1}{2}(1-dtdxdy)=\epsilon^{-}\tau^{+}, \tag{49}$$

so that we do not get new idempotents and thus not an extended split.

We could keep trying to find some other (pair of) idempotent(s) to multiply $\epsilon^{\pm}\tau^{*}$ and which would commute with them. None exists. We then say that the $\epsilon^{\pm}\tau^{*}$ are primitive as they do not comply with the following definition. An idempotent is said to be primitive if it cannot be decomposed into a sum $A+B$ of two commuting, mutually annulling idempotents, i.e. such that $AB = BA = 0$. Since the dimension, 4, of spacetime is rather low for present purposes, one can readily find by trial and error that the $\epsilon^{\pm}\tau^{*}$ are primitive. For higher dimension one resorts to a so called Radon-Hurwitz theorem to find the number of such idempotents as a function of dimension and signature of the metric. A detailed exposition of this subject does not pertain here and would also take too much room. We refer interested readers to the book "Clifford Algebras and Spinors" by Pertti Lounesto. For the case in point, application of the theorem confirms that the $\epsilon^{\pm}\tau^{*}$ are primitive. To be clear as to the meaning of the theorem, let us say that there are more primitive idempotents in 3-D Euclidean space and in spacetime, like the $\frac{1}{2}(1+dx)\frac{1}{2}(1*idydz)$, but this is not a split that extends the one by $\epsilon^{\pm}\tau^{*}$. For our purposes, the Radon-Hurwitz theorem has to do with the process of continuing to split idempotents.

3 Application 1: Theorems of residues and Cauchy's

This section is a typical one on applications, i.e. one where the application is completed. In the KC , we distinguish two parts in what goes by the name of calculus of complex variables as taught in an undergraduate course in the subject. There is a major difference between the two. It is for this reason that we treat the subject in two different chapters, the two treatments in the sense that one does not require the other. They are two different topics. In this application, we do not need a concept of differentiation additional to the one in any course on the real calculus of several variables. Algebra suffices, namely the algebra that we have just seen in section 1. We shall perform certain real integrals for which the standard calculus of complex variable is typically used. Instead of the complex plan, we shall have a Clifford algebra $Cl_{(0,1)}$ of differential forms viewed as even subalgebra of $Cl_{(2,0)}$. We intend to present her a polished and more comprehensive version of the presentation of the theorems of residues and of Cauchy presented in a paper in arXiv. Just type Jose G Vargas on the top right hand corner and scroll a little bit down to find the pdf file of the paper "Real Calculus of Complex Variable: Weierstrass Point of View". Since readers already have this available to them, we temporarily skip the writing of this section. We shall thus use our time to get to the core of the Kähler calculus as fast as possible.

In an application in the next chapter, on the other hand, we begin the representation of the theory of complex variable without complex variable. It can deal in principle with any problem which does not even exist in standard real analysis, although it will be focussed on integrals because of the interest of the targeted audience. We shall thus define integrations which, in terms of differential forms, would put a differential

2−form as integrand of a line integral. Of course, this is a totally new game. What will correspond to an integral with complex integrand will require some new concepts in calculus with real differential forms.

4 Application 2: Algebraic template for concepts like collapse, entanglement, confinement and teleportation

All the physical applications of the KC revolve around the split

$$1 = \epsilon^+\tau^+ + \epsilon^+\tau^- + \epsilon^-\tau^+ + \epsilon^-\tau^-, \tag{50}$$

or the still simpler splits considered below. This particular one is related to the pair spin and rest mass, as we shall learn in the electromagnetic environment. Neither the electromagnetic differential 2−form nor the electromagnetic potential are members of any of the four ideals defined by the four idempotents $\epsilon^\pm\tau^*$, but certainly can be decomposed into members of them. Kähler showed in an argument in his 1961 paper —complemented with work in 1962 on charge and antiparticles— that electrons and positrons of both chiralities relate to those idempotents in a one to one correspondence. These play an even larger role than the phase factors —which also are essential— in determining the treatment of solutions of quantum mechanical equations involving particles.

In the Dirac theory, idempotents do not play the core role that they play in the Kähler calculus for decomposing wave functions that do not belong to an ideal. The reason is that the Dirac equation is ab initio about elements of ideals, and the Kähler equation is about members of the whole algebra; the members of the ideals are but a very important development. This is just but one of the reasons why Kähler's quantum mechanics supersedes Dirac's.

The split (50) will later be extended to more comprehensive splits, which he did not pursue. For that, he first should have geometrized the imaginary unit, its role then being played not only by $dxdy$, but also by $dydz$ and $dzdx$, all three simultaneously. But, at this point, we would already be outside the realm of algebra and calculus with scalar-valued differential forms. The extension of (50) would take place through primitive idempotents consisting of three factors, which include one each of the pairs $\epsilon^\pm$ and $\tau^\pm$. We shall later explain how this is possible.

For the moment, let us go into what (50) has to offer. This split implies that any element of the algebra can be written as a sum

$$u = \ {}^+u^+\ \epsilon^+\tau^+ + \ {}^+u^-\ \epsilon^+\tau^- + \ {}^-u^+\ \epsilon^-\tau^+ + \ {}^-u^+\ \epsilon^-\tau^-. \tag{51}$$

This decomposition is unique if we demand that the coefficients ${}^\pm u^*$ are elements of the Kähler algebra that depend on $d\rho$ and dz but not on $d\phi$ and dt. This demand does not entail lack of generality.

Since the four idempotents $\epsilon^{\pm}\tau^{*}$ are mutually annulling, we multiply this equation by $\epsilon^{+}\tau^{+}$ on the right and obtain

$$u\epsilon^{+}\tau^{+} = \ {}^{+}u^{+}\ \epsilon^{+}\tau^{+}, \tag{52}$$

and similarly for products of (51) on the right with the other three idempotents. We find the ${}^{+}u^{+}$ by performing operations in (52) that move the factors dt and $d\phi$ in u to the right to be absorbed by the idempotents. In computations of later chapters, we shall encounter examples of these absorptions. But how can these ideals represent leptons?

The ideals represented in (51) through an arbitrary element of the algebra are intimately connected with the form that solutions with time translation and rotational symmetry take. The association of $\epsilon^{\pm}\tau^{*}$ with electrons and positrons is but a first step in the association of ideals with particles, including muons, taus and quarks. We also need a corresponding phase factor. The $\epsilon^{\pm}$ idempotents take care of the dependence on dt and the corresponding phase factor then takes care of the dependence on t, which has to do with differentiation, not with algebra. We then need the concept of constant differentials (chapter 3). The $\epsilon^{\pm}\tau^{*}$'s are constant differentials. This has the implication that differentiation of a member of any of those ideals remains in the ideal. This propagates to the Kähler equation (chapter 4), which may then be viewed as an entanglement of four equations. Solutions will correspond to actual leptons when some external factor like an electric field or a measuring instrument produces the *collapse* of that entangled system. Until that happens, the spinors are not necessarily particles. Notice the expanded meaning that collapse has here. What is additional in Kähler's relative to Dirac's quantum mechanics is that we have a richer variety of superpositions because the superimposed elements are entangled. Teleportation phenomena are a case of collapse in this extended sense. The Cartan-Kähler extension of Kähler theory will provide us with a better tool kit to interpret what is it that travels in opposite directions and that ends in a correlated collapsed that apparently violates causality. A violation does not actually occur, as we shall explain further below.

The simpler split

$$u = \ {}^{+}u\ \epsilon^{+} + \ {}^{-}u\ \epsilon^{-}. \tag{53}$$

already has the major implication of yielding a concept of charge which comes in types positive and negative, but both for the same sign of the energy. This result will apply in particular to the ideals generated by $\epsilon^{\pm}\tau^{*}$ and, more particularly, to pair creation and annihilation. The emergence of negative energy solutions in Dirac's theory is an spurious effect. Both particles and antiparticles are in the same footing.

The split

$$u = u^+ \tau^+ + u^- \tau^- \tag{54}$$

also is deeply involved with the foundations of quantum mechanics. Not only do orbital and spin components of angular momentum come together, but they actually are born together, not as twins, but as non-invariant terms which only become invariant after we take something from one of them and add it to the other. Spin is as external as orbital, and orbital is as internal as spin. This confirms at a very profound level that the concept of particle emerges from the concept of field solutions of basic equations. The field is not exchange currency (quantized or not) among merchants (particles).

The use of Clifford algebra to deal with rotations makes into bivectors the imaginary units in the exponents of the phase factors. Because of the intimate correspondence between idempotents and phase factors in solutions with symmetry of exterior systems, the same observation applies to the imaginary units in their associated idempotents. The $\epsilon^{\pm}\tau^{*}$'s then become $\epsilon^{\pm}I^{*}_{ij}$, where $I^{\pm}_{ij}$ is $\frac{1}{2}(1 + \mathbf{a}_i\mathbf{a}_j dx^i dx^j)$ with no sum over repeated indices. It is then natural to see here three different generations of leptons. Spontaneously broken anisotropy of 3-space is needed to justify this difference, three special directions being associated with three generations. For this to be consistent with experiment, the Lorentz transformations must remain physically relevant. They do, this consistency having been known to philosophers of science (Reichenbach, Grünbaum) and physicists like David Bohm (see his book on special relativity). But much more remains to be done. Because of the possibilities opened by this compatibility, something like the weak interactions appears to be present in Kähler's quantum mechanics with geometrized imaginary unit.

Consideration of anisotropy is not an ad hoc assumption for particular programs by physicists, though it may have been so at some time. It is brought to the fore by the evolution of the theory of connections, as explained in our $U(1) \times SU(2)$ paper (google "$U(1) \times SU(2)$ from the tangent bundle" to find and freely download it).

The aforementioned geometrization of the imaginary unit leads to a canonical Kaluza-Klein space, where there are obvious classical and quantum sectors associated with two 4−D subspaces. In the classical sector, the constant speed of light reigns supreme. But there is no such obvious limitation in a purely quantum sector, where information within a pure field configuration u might travel as superluminal speeds. I said might, not may, since we do not know better at this point; but this is enough to start looking at the issue of teleportation in a new light.

The connection of the last three paragraphs with the split (50) is that we have three copies of the same, but with each of the three independents for each copy.

We finally develop the consequences of the existence of a commutative substructure in the Cartan-Kähler structure. It looks like a Clifford algebra of sorts, but is not so because it is commutative. Nevertheless products in two Clifford algebras are involved in these commutative products. Since idempotents involve only mirror elements (dx mirrors $\mathbf{i}$, but dy does not) there is an additional extension of the split (51) by virtue of this commutativity. For brevity reasons, we shall again speak of the newly relevant idempotents instead of speaking of the split itself. The latter involve three idempotent factors, extending the $\epsilon^{\pm} I^{*}_{ij}$ with space translation ones, which thus come in triples, one for each generation (It might look as six, but there is redundancy when the products are actually performed). The signature, however, does not allow for decreasing exponentials, i.e. for phase factors. This implies that the particles that would correspond to these idempotents do not reach very far; they start dying as they are born. There can never be enough energy if the wave function does not decrease with distance, much less if it increases. Call this confinement. It will look as a surface effect, not without reason. Translational symmetry when matter is involved must include surface effects, since the symmetry does not extend to infinity. Confinement follows.

Much of what has been said in this section may be viewed as speculation. It certainly is. Speculation stars to become theory as it gains ever more sophisticated mathematical representation. The bridge between theory and experiment is very large, as Einstein said. No single person can build alone the large bridges that the advanced state of the physics makes necessary when its foundations are concerned. But how would you interpret the mathematics we have developed in the hypothetical physical situations to which this mathematics might apply.