CHAPTER 4: Relativistic Quantum Mechanics

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Once again, lecturers will choose from this chapter what they will teach on day 6 of the summer school.

This chapter was going to be called Kähler equations. I changed the title when I recently learned that a group of billionaires (like Ambani, Bezos, Branson, Gates, Ma, Soros, Zuckerberg, etc. is launching a major clean energy initiative (For fair treatment of those that I have not mentioned, please go breakthroughtenergycoalition.com/en) Energy is an issue for exceptionally good theoreticians, like Einstein in the first half of the twentieth century (energy of photons, energy equals mc^2, the energy-momentum tensor equals the Einstein tensor) and Julian Schwinger (See the last section of this chapter for more on Schwinger, the other Feynman or even better) in the second half. The ideas of both, Einstein and Schwinger, in the last decades of their lives were dismissed. In the last section, we also say a little bit about the connection between Kähler's calculus and Einstein's failed at unification with teleparallelism.

Kähler's quantum mechanics (actually, the Kähler equation by itself) has to do with the initiative of the coalition, even though its proponents do not know it. It prompts and promises a better understanding of energy at the quantum level. Present day quantum physics is being built with inadequate arguments, which are at the root of the spurious emergence of negative energy solutions in Dirac's theory. It is resolved by means of so called "hole theory", where all states with negative energy are filled and thus constitutes a sea of infinite energy density. Holes (or lack of negative energy solutions) are interpreted as positrons (A very good book for all of this is "Relativistic Quantum Mechanics" by J. D. Bjorken and S. D. Drell, both from the Stanford Linear Accelerator Center"; its chapter five is about hole theory). The infinite energy density of the sea is reduced to still huge finite ones through a couple of independent artifacts which do not even give the same results (They actually

give results that differ by dozens of orders of magnitude). In contrast, Kähler's equation entails the emergence of positrons with the same sign of energy as electrons. Since we reserve that derivation for chapter six, we send impatient readers to the very end of section 26 of Kähler's 1962 paper "Der innere Differentialkalkül".

Also recently —while writing these chapters— I heard again the often repeated statement that quantum physics and general relativity are mathematically and conceptually incompatible. But, with the Kähler calculus, the mathematics of general relativity are contained in the mathematics of quantum mechanics. They are not, therefore, mathematically incompatible. As for conceptual incompatibility, it is not so because of the nature of the subject but because of the Dirac equation. In his time, it was a tremendous conquest, but provisional and not quite as intelligible as physics used to be). Its limitations led to misinterpretations as to the true nature of the field. If, as this author claims, the Kähler calculus is the right equation, the quantum mechanics that ensues is more about a field in the classical mold rather than about particles and probability amplitudes. The latter (certainly as well as their square) should be viewed as derived or emergent concepts. But does one really believe that the mathematics for both sectors of physics they will incompatible for ever? We should not replace "we do not know how to do ..." with "it cannot be done". And, fortunately, Kähler's version of quantum mechanics is such that the day to day practice of most of quantum physics will not change; only the foundations will.

In order to provide an early glimpse of Kähler's approach to quantum physics, we shall alter for the summer school what should have been the order of chapters in the presentation of the theory. In this chapter, we shall mention, though only briefly, Kähler's obtaining of the strict harmonic differentials in 3-D Euclidean space (minus the origin), where they coincide with the harmonic differentials. Such a study should logically follow the study of rotational symmetry (chapter 5), which is instrumental in solving the fine structure of the hydrogen atom. We shall leave for readers to follow the long calculations directly from the Kähler papers, which they should be able to read with little knowledge of German after we have provided here the basic foundations. Chapter 6 will deal with the conservation law and related applications (the two types of charge, a uniqueness theorem for integration of Helmholtz type systems, etc.).

Starting with chapter 7, I would present my own work on the Cartan-Kähler calculus of differential forms, where physics and mathematics look as if they were the same thing. Fortunately for participants in this summer school, you will not have to suffer me much longer since, by then, it will be the early morning of the last day of the school.

What does all this have to do with the breakthrough energy coalition? A better platform for quantum mechanics is a better platform for understanding energy. One could bring the Kähler calculus —and thus quantum physics and the understanding of energy— to greater heights with an amount of funds that would be pocket money for billionaires, specially if the funds are used in countries where eager scholars draw meager salaries. I have written this for the record so that one day, perhaps not too far for me to see it, mathematically gifted acquaintances of those billionaires let them know of this opportunity to carry on their project with the much more sophisticated Kähler theory. It may lead to less costly and more efficient and environmentally friendly way of obtaining clean energy. Schwinger may have pointed the way with his source theory, and Kähler mathematics is the right tool to make his theory more accessible.

1 Kähler equations

With the explicit purpose of accelerating the bringing about of a more complete picture of what the Kähler calculus has to offer, I am starting to skip details of computations present in Kähler's papers.

1.1 Solutions with time symmetry of exterior systems

The form of solutions with symmetry of exterior systems is independent of the system itself. We mean what types of factors represent the symmetry in solutions that have it. The dependence on coordinates that are parameters of symmetries is given by a phase factor, and the dependence on their differentials is given by idempotents that are constant differential forms.

Recall a decomposition such as

$$u = {}^{+}u\,\epsilon^{+} + {}^{-}u\,\epsilon^{-}. \tag{1.1}$$

It corresponds to time translation symmetry. The phase factor must, therefore, be e^{-iEt}. As we shall son start to understand and with $\hbar = 1$, a negative charge solution with time translation symmetry should be given by

$$e^{-iEt}R(x, dx)\epsilon^{-}. \tag{1.2}$$

By virtue of the fact that the $\epsilon^{\pm}$ are constant differentials, we shall have equations

$$\partial u = \quad (\partial\ {}^{+}u)\ \epsilon^{+} + (\partial\ {}^{-}u)\ \epsilon^{-}, \tag{1.3}$$

implicit in the more comprehensive equation

$$\partial u = \ (\partial\ {}^{+}u^{+})\ \epsilon^{+}\tau^{+} + (\partial\ {}^{+}u^{-})\ \epsilon^{+}\tau^{-} + \ (\partial^{-}u^{+})\ \epsilon^{-}\tau^{+} + \ (\partial^{-}u^{+})\ \epsilon^{-}\tau^{-} \tag{1.4}$$

for when, in addition to time translation symmetry, one also has rotational symmetry (See chapter 2 for all these idempotents). Suffice to rewrite this equation as

$$\partial u = \ [(\partial\ {}^{+}u^{+})\ \tau^{+} + (\partial\ {}^{+}u^{-})\tau^{-}]\ \epsilon^{+} + [(\partial^{-}u^{+})\ \tau^{+} + (\partial^{-}u^{+})\tau^{-}]\epsilon^{-} \tag{1.5}$$

and then as

$$\partial u = \ \partial({}^{+}u^{+}\ \tau^{+} + \ {}^{+}u^{-}\tau^{-})\ \epsilon^{+} + \ \partial({}^{-}u^{+}\ \tau^{+} + \ {}^{-}u^{+}\tau^{-})\epsilon^{-} \tag{1.6}$$

$\tau^{\pm}$ has to do with issues that come under the name of handedness, chirality and helicity, intrinsically associated with spin. This will be understood when, in the next chapter, we deal with angular momentum.

For a system where the rest mass of the electron were the dominant energy, it makes practical sense to write E as $m+\Delta E$. This is well known from standard relativistic quantum mechanics. In the Kähler calculus, the dominant energy representation of electron systems is given us

$$u = e^{-imt}R(t, x, dx)\epsilon^{-}, \tag{1.7}$$

where $R(t, x, dx)$ depends the more slowly on time the more close to each other m and E are. Compare with (1.2). This expression for solutions will be very much used in the following.

1.2 Kähler equations for stationary solutions

Let u be any element of the Kähler algebra (again, of scalar-valued differential forms). Any equation of the form

$$\partial u = au \tag{1.8}$$

will be referred to as a Kähler equation. A particular case is $\partial u = 0$. In Euclidean space of dimension 2 we get even differential forms, already considered in the two previous chapters. In dimension 3, Kähler derived a magnificent theory of harmonic differentials without resort to separation of variables. That is very interesting but momentarily, it is not as important as the equation (1.8) in spacetime, with a different from zero and the algebra being over the complex field. We shall also make $c = 1$.

$$(\partial^{\pm}u)\epsilon^{\pm} = \ a\ ({}^{\pm}u\ \epsilon^{\pm}). \tag{1.9}$$

Solutions ${}^{\pm}u\ \epsilon^{\pm}$ may correspond to particle states of undefined chirality.

From the perspective of the number of components in the solutions and what they represent, neither Eq. (1.8) nor Eqs. (1.9) have a close analog in the Dirac equation. To start with, they do not have the same number of components. The electron and positrons of both chiralities are directly associated with the equations

$$(\partial\ {}^{\pm}u^{*})\epsilon^{\pm}\tau^{*} = \ a\ ({}^{\pm}u^{*}\ \epsilon^{\pm}\tau^{*}). \tag{1.10}$$

The parentheses have been put there simply to emphasize that the differentiation of $\epsilon^{\pm}\tau^{*}$ can be ignored since it yields zero. The asterisk means that correlation between the superscripts of ϵ and τ is not implied. The $\epsilon^{\pm}$ would be for pairs of particle and antiparticle, and the $\tau^{\pm}$ would be for the two opposite handedness in each of those two cases. Phase factors are inside the ${}^{\pm}u^{*}$ when solutions proportional to ${}^{\pm}u^{*}\ \epsilon^{\pm}\tau^{*}$ represent specific particle states.

By writing a as $\alpha + i\beta dt$ and replacing ${}^{\pm}u$ with $e^{-iEt}p^{\pm}$, the development of Eqs. (1.9) by Kähler yields the following equations for stationary solutions

$$\partial p \pm (E+\beta)\eta p\ - \alpha p = 0, \tag{1.11}$$

where the signs $\pm$ corresponds to $p^{\pm}$ respectively. This is obtained without resort to dominant energies or specific couplings. Kähler used this equation just for the fine structure of the hydrogen atom, for which it takes a very simple form. We shall later see another important application.

1.3 Kähler equations for Dirac spinors

We may be interested in what we shall denote as Dirac spinors that are proper "functions" for given energy and chirality. Kähler wrote such spinor solutions as

$$u = e^{is\phi - iEt}p\tau^{\pm}\epsilon^{*}, \tag{1.12}$$

where p depends only on ρ, z, $d\rho$ and dz, and where s and E are the angular momentum and energy of the system. We shall replace the symbol $\vee$ for juxtaposition when we judge that it makes equations more transparent.

Differentiating (1.12), we get

$$\partial u = e^{is\phi - iEt}(isd\phi \vee p - iEdt \vee p + \partial p)\tau^{\pm}\epsilon^{*}. \tag{1.13}$$

We then use

$$d\phi \vee p = d\phi \wedge p = \eta p \wedge d\phi = \eta p \vee d\phi, \tag{1.14}$$

and

$$dt \vee p = dt \wedge p = \eta p \wedge dt = \eta p \vee dt. \tag{1.15}$$

Thus,

$$\partial u = e^{is\phi - iEt}(is \vee \eta p \vee d\phi - iE\eta p \vee dt + \partial p)\tau^{\pm}\epsilon^{*}. \tag{1.16}$$

We now use that

$$\tau^{-} - \tau^{+} = -idxdy = -i\rho d\rho d\phi \tag{1.17}$$

to obtain $d\phi$ and then

$$d\phi \vee \tau^{\pm} = -i\frac{d\rho}{\rho}(\tau^{-} - \tau^{+})\tau^{\pm} = -i\frac{d\rho}{\rho}\tau^{\pm}. \tag{1.18}$$

Equation (1.18) together with $idt\epsilon^{\pm} = \mp\epsilon^{\pm}$ allows us to finally write (1.16) as

$$\partial u = e^{is\phi - iEt}(\partial p \pm s\eta p\frac{d\rho}{\rho} - *E\eta p)\tau^{\pm}\epsilon^{*}, \tag{1.19}$$

which could be used on the left hand side of Kähler equations, specially if we do not want to resort to dominant energies in reducing the simplicity of some Kähler equations.

2 Dominant energy and electromagnetic coupling

In the first subsection of this section as well as everything that went above is original work of Kähler. If you do not like it, go to church and talk to him. From now on, I take the blame for what follows. I say that because some learned people will retort that this is not in the spirit of modern physics and mathematics, or they will say even something worse. I do not care. The mathematical and physics community should at least know what Kähler did. He wrote in German. I am doing it in English and in a slightly more Cartanian way. One has to understand geometry à la Cartan in order to really appreciate the Kähler's calculus and even go beyond it. With my book "Differential geometry for physicists and mathematicians" I tried to do with some of Cartan's work what I am now trying to do with some of Kähler's work.

2.1 Charge

With electromagnetic coupling, the Kähler equation reads

$$\partial u = (-m + ieA)u, \tag{2.1}$$

where m is the rest mass, and e (positive or negative) represents the charge of positrons and electrons. In going from (1.8) to (2.1), there are unstated or undefined assumptions. We are trying to say that, whereas (1.8) does not assume a situation where particles and antiparticles are of

the essence, (2.1) does, even though this is not explicit. It is contained in the mere presence of a mass competing with an electromagnetic field. For the split (1.1),we have

$$(\partial^{\pm} u)\ \epsilon^{\pm} = (-m + ieA)\ {}^{\pm}u\ \epsilon^{\pm}. \tag{2.2}$$

Let us now deal with the announced existence of what, in the electromagnetic case, becomes charge of opposite signs. We shall reproduce in chapter 6 the derivation of the conservation equation of this calculus and its specialization to when u is written as (1.1). After the appropriate natural interpretations, the conservation law takes a form which translated to the language of the vector calculus reads like this

$$\left(\frac{\partial \rho_+}{\partial t} + div\ j_+\right) + \left(\frac{\partial \rho_-}{\partial t} + div\ j_-\right) = 0, \tag{2.3}$$

where ρ_+ and j_+ are determined only by ${}^{+}u$ and its complex conjugate, and similarly for ρ_-, j_+ and j^-. Hence the two options in $\epsilon^{\pm}$ should be assigned to positrons and electrons. Probability densities should in turn be assigned directly to charge and indirectly to particles, but certainly not yet at this point.

We shall not go until chapter 6 into the rather laborious derivation of how $\rho_{\pm}$ and $j_{\pm}$ depend on ${}^{\pm}u$. Equation (2.3) implies the apparent existence of two types of charge, and that only total charge (understood as their sum) and not each individual one is conserved when both are present. Their individual conservation is an afterthought when the two types stay apart, like a nucleus and its surrounding cloud of electrons.

In obtaining (2.3), Kähler does not make full use of the Kähler equation with electromagnetic coupling, but simply that a for that coupling has a specific property, which can also be satisfied in principle by other ones. And even if that property were not satisfied, we might still get conservation equations with the flavor of (2.3), but where the ρ's and j's would depend on the ${}^{\pm}u$'s in a different way. Results like this do not make part of the books on the Dirac theory from which I, and probably you, learned.

The equations we derived in the previous section can easily be adapted to the electromagnetic coupling. We are about to see that the equations for positrons and electrons have nothing to do with small and large components of the solutions of one equation, solutions misinterpreted in the Foldy-Wouthuysen treatment of Dirac's theory as respectively representing positron and electron amplitudes. Large and small components belong equally to solutions for particles and antiparticles. These two are in the same footing at this level of theory.

2.2 The Pauli equation

Of practical interest is the case when the rest energy m of the electron is the dominant energy. We use (1.7) in (2.1) for an electron in a state of undefined handedness in an electromagnetic field. Given that ∂u is $dt\ u_{,t} + dx^i\ u_{,i}$, we get, after some simplification,

$$\left[dt(-imR + R_{,t}\,) + dx^i R_{,i}\,\right]\epsilon^- = (-m - ie\Phi dt + ieA_i dx^i)R\epsilon^-. \quad (2.4)$$

Premultiplying by $-dt$, using that $(dt)^2$ is -1 and leaving $R_{,t}\,\epsilon^-$ as the sole term on the left, we get

$$R_{,t}\,\epsilon^- = \left[-dx^i dt R_{,i} + (-ieA_i dx^i dt + im + mdt - ie\Phi)R\right] dt\epsilon^- \quad (2.5)$$

We cannot simplify ϵ^- until we absorb all dt factors into ϵ^-, which we do by means of $dt\epsilon^- = -i\epsilon^-$ (Check what would happen and compare with what we are about to get). We thus obtain

$$R_{,t} = -P\eta R - ie\Phi + im(R - \eta R), \quad (2.6)$$

where we have replaced $dx^j(-i\partial_j - eA_j)$ with the symbol P and where we have used that R does not contain dt as a factor. Recall from previous chapters that η changes the sign of terms of odd grade. We apply η to (2.6). Combining the resulting equation with (2.6) itself, we get

$$\varphi_{,t} = P\chi - ie\Phi\varphi, \quad (2.7)$$

$$\chi_{,t} = -P\varphi - ie\Phi\chi + 2im\chi, \quad (2.8)$$

where

$$\varphi = \frac{1}{2}(R + \eta R), \qquad \chi = \frac{1}{2}(R - \eta R). \quad (2.9)$$

We now proceed following a step to be found in the standard literature. Among the books on relativistic or advanced quantum mechanics that we have come across, there is one which deserves special attention because of the topics it deals with, namely the aforementioned "Relativistic Quantum Mechanics" by J. D. Bjorken and S. D. Drell". This book is particularly interesting not only because of the institutional affiliation of its authors and the contributions they made to the high energy physics of their time, but also because of the year in which their work was published, 1963. That was the time when high energy physics was becoming corporate physics, meaning that advances were produced as if they were products of a corporation, one of the main headquarters being Stanford's linear accelerator. It was also the time when problems in the very foundations of quantum mechanics were addressed rather than swept under

the rug, as, more often than not happens nowadays. This is why there was a chapter on hole theory in their book. We have mentioned all this because we are about to take a step right now which is conceptually the very same step as one of those that they took in their first chapter. In the paragraph just before the last, they had connected with the Pauli equation. They then opened the last paragraph of the chapter as follows: "Fortified by this successful non-relativistic reduction of the Dirac equation, we go on ..."

The title of their chapter 4 is "The Foldy-Wouthuysen transformation". And chapter five's title is "Hole theory", designed to deal with the issue of negative energy solutions of Dirac's theory. These solutions are a consequence of what is spurious in Dirac's theory. The association of antiparticles with negative energy solutions does not emerge when doing relativistic quantum mechanics with Kähler's mathematics.

In the process of deriving the Pauli equation, Bjorken and S. D. Drell took a step which is totally equivalent to the one that we are about to take with our equation (2.8). Of their equation that parallels this equation (2.8), they said that "it may be approximated for kinetic energies and field interaction energies small in comparison with $mc^2, ...$". We return to this further below, where we shall see that this amounts to neglecting terms $\chi_{,t}$ and $-ie\Phi\chi$ in our equation (2.8). The assumption $\chi_{,t}$ is unwarranted, since $\chi_{,t}$ may be huge and can thus not be ignored even if $\chi_{,t}$ were small. This will have a very important consequence that is worth mentioning with a view to trying to understand at least some obscure point in the Dirac theory, as we shall explain further below. Let us assume that we ignore this difficulty and proceed in parallel to what they did. If the said approximation were correct, we would have

$$\chi_1 = -\frac{i}{2m}P\varphi, \qquad\qquad i\varphi'_{,t} = iPP\varphi + i\Phi\varphi. \tag{2.10}$$

Let us assume that we knew φ. We would then approximately know χ. We refer to this approximated χ as χ_1. With χ replaced with χ_1, the right equation (2.7) gives only an approximate $\varphi_{,t}$ even if we knew the right φ and left it on the right. This is the reason why we have used φ' on the left and φ on the right. This remark will be useful for the approximation in the next section

We expand $PP\varphi$ and use that

$$P_kP_j\varphi = -i\partial_k(ieA_j) = ieA_{j,k}\,. \tag{2.11}$$

We then perform some manipulations given in more detail in our "The Foundations of Quantum Mechanics and the Evolution of the Cartan-

Kaehler Calculus" (Foundations of Physics **38**, 610-647 (2008)), and get

$$i\frac{\partial}{\partial t}\varphi' = \frac{1}{2m}P^2\varphi + \frac{ie}{2m}B_k dx^i dx^j \varphi + e\Phi\varphi, \tag{2.12}$$

where the summation is over the three cyclic permutations of $(1, 2, 3)$. This is the Pauli equation in terms of differential forms, except that Pauli's φ on the right hand side also is φ'. At this stage and for the present purpose, "this difference does not make a difference".

One should not overlook that this equation is for φ, not for R, here as in the corresponding formula in the Dirac theory. In spite of the objection raised about (2.10), equation (2.12) is of great interest,. though not for the same reason as in the paradigm, where it was welcome because it gave confidence in the Dirac equation to construct a relativistic theory of the electron. For us, it will be important for another reason, which we shall discuss at length later on.

2.3 The Foldy-Wouthuysen Hamiltonian

In the literature, one uses the term Foldy-Wouthuysen transformation more often than Foldy-Wouthuysen transformation. But its main purpose is the transformation of the Hamiltonian, which Bjorken and Drell perform in the every same section. In fact, they do not even speak of the transformed wave function on which the new Hamiltonian acts, though this is obvious by gauge invariance considerations.

We continue the process by which (2.12) was obtained. We write the sought χ as the old one, χ_1, plus some χ_1',

$$\chi = -\frac{i}{2m}P\varphi + \chi_1'. \tag{2.13}$$

This together with (2.8) yields

$$\chi_{,t} = -ie\Phi\chi + 2im\chi_1'. \tag{2.14}$$

Hence, solving for χ', we have

$$\chi_1' = \frac{i}{2m}\chi_{,t} + \frac{e\Phi}{2m}\chi. \tag{2.15}$$

From (2.7), we get that χ_1' adds the terms $iP\chi_1'$, i.e.

$$iP\chi_1' = \frac{-1}{2m}P\chi_{,t} + i\frac{e\Phi}{2m}P\chi. \tag{2.16}$$

It is only at this point that we replace χ with χ_1 on the right hand side of (2.16). The computations are straightforward but rather laborious. One obtains

$$iP\chi_1' = \frac{-1}{8m^3}p^4\varphi - i\frac{e}{4m^2}PE^C\varphi - \frac{e}{4m^2}E_{i\,,\,j}dx^j \wedge dx^i\varphi, \tag{2.17}$$

where we have eliminated higher order terms for comparison purposes with how this would be written in the paradigm. The small p is $-idx^j\partial_j$, and E is here notation for dx^jE_j. The superscript is used to indicate that E in $PE^C\varphi$ is treated as if it were a constant in this term. The parenthesis around $dx^j \wedge dx^i$ is meant to say that we add over cyclic permutations in $1^k dx^i \wedge dx^j$.We thus further get

$$-\frac{ie}{4m^2}PE^C\varphi = -\frac{e}{4m^2}dx^i dx^j E_j \partial_i \varphi, \tag{2.18}$$

which corresponds to the $\sigma \cdot (\mathbf{E} \times \mathbf{p})$ term in the standard version of relativistic quantum mechanics, i.e. with vector-valued quantities. Please be informed that we had an inadvertent replacement of a "$\vee$" product with a "$\wedge$" product. That in turn gave rise to a spurious term, which is the reason for a discrepancy with (2.18).

The last term in (2.17) yields

$$-\frac{e}{4m^2}E_{i\ ,\ j}dx^j \vee dx^i\varphi = \frac{e}{4m^2}E^i{}_{,i}\ \varphi - \frac{e}{8m^2}E_{j\ ,i}dx^j \wedge dx^i\varphi, \tag{2.19}$$

equivalently, the $div\mathbf{E}$ and $\sigma \cdot curl\mathbf{E}$ terms with appropriate factors. In all terms on the right of the $i\frac{\partial}{\partial t}\varphi'$ post-Pauli equation, all operators are acting on φ the original wave differential form φ.

Let us look at what we have got. We have developed to order $1/m^3$ the Kähler equation with electromagnetic coupling and mass as dominant energy term for the even grade part of solutions whose odd grade part satisfies $\chi_{,t} \ll -P \vee \varphi$. The development was directed towards letting $i\frac{\partial}{\partial t}\varphi$ alone on the left hand side. Hence, the right hand side defines a Hamiltonian, namely

$$\begin{aligned} H = &\frac{1}{2m}P^2 + \frac{ie}{2m}B_k dx^i dx^j + e\Phi - \frac{1}{8m^3}p^4 - \frac{e}{4m^2}dx^i dx^j E_j \partial_i \\ &+ \frac{e}{4m^2}E^i{}_{,i} - \frac{e}{8m^2}E_{j\ ,i}dx^j dx^i, \end{aligned} \tag{2.20}$$

acting on the original wave function, whereas the energy operator, $i\partial/\partial t$ is acting on a slightly different wave function, i.e.

$$i\frac{\partial}{\partial t}\varphi' = H\varphi. \tag{2.21}$$

Of course, we could think of φ' as a φ, and vice versa, thus $i\frac{\partial}{\partial t}\lambda = H\lambda'$, where we have used λ's instead of φ's in case it would create confusion.

Observe how, by virtue of the little bit of Kähler equations that we have learned, we have reached deep into relativistic quantum mechanics, to a depth even greater than most advanced books on quantum mechanics.

2.4 Leftovers

TO BE COMPLETED. This subsection would deal with subjects that we left over because of the pressure for this author to write about other jewels of this calculus and concomitant quantum mechanics.

Readers can now do exactly the same for positrons. One only has to repeat with

$$u = e^{-imt}R(t, x, dx)\epsilon^{+}, \tag{2.22}$$

what we did with $e^{-imt}R(t, x, dx)\epsilon^{-}$. You will find a system just like (2.7)-(2.8), except that the even, $\bar{\varphi}$, and odd, $\bar{\chi}$, parts of u will satisfy the equations satisfied by χ and φ. Thus $\bar{\varphi}$ will be small and $\bar{\chi}$ will be large under similar assumptions. But notice that both χ and φ belong to the electron and both $\bar{\varphi}$ and $\bar{\chi}$ belong to the positron. Whether we have one particle or another depends only on whether we are dealing with ϵ^{+} or ϵ^{-}.

Another topic that will be treated in this subsection is the explanation from the Kähler theory of why the mass of the positron emerges as $-m$. In this and following chapters, I shall provide you with entries into the world of Kähler, exploring areas that nobody has jet explored.

3 An entry point for research on relativistic quantum mechanics with the Kähler calculus

What follows might be felt as being derogatory on Dirac, whom many will rightly consider the second best physicist of the twentieth century. His equation presently is more relevant for theoretical physics than Einstein equations. Dirac did his work when the math was not ready for the task. So, thanks are due to Dirac. But now is now and the mathematics allows for a better job. We proceed to describe the situation with respect to post-Pauli-Dirac electromagnetic Hamiltonians. For the discussion that will follow, we write the Dirac equation with explicit universal constants

$$i\hbar\frac{\partial\psi}{\partial t} = \left[c\alpha\cdot\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right) + \beta mc^2 + e\Phi\right]\psi, \tag{3.1}$$

where α and β represent well known matrices in relativistic quantum mechanics. Clearly, a development in powers of $1/m$, which is what Bjorken and Drell did, is not equivalent to a development in powers of $1/c$. The reason to make this remark will be seen later below.

3.1 Post-Pauli-Dirac and Foldy-Wouthuysen equations

The term Foldy-Wouthuysen Hamiltonian is not used in the literature. One uses instead the term Foldy-Wouthuysen to refer to transformations through which one reaches the Hamiltonian:

$$H = \beta\left(m + \frac{(\mathbf{p} - e\mathbf{A})^2}{2m} - \frac{\mathbf{p}^4}{8m^3}\right) + e\Phi \; - e\frac{1}{2m}\beta\sigma\cdot\mathbf{B}$$
$$- \frac{ie}{8m^2}\sigma\cdot\, curl\ \mathbf{E} \; - \frac{e}{4m^2}\sigma(\mathbf{E}\times\mathbf{p}) \; - \frac{e}{8m^2}div\mathbf{E}, \tag{3.2}$$

where $\mathbf{p}$ has components p^i, where Φ and $\mathbf{A}$ are components of the 4-vector potential and where σ has as components the three Pauli matrices. Equation (3.2) results in the approximation (kinetic energy/m)^3 and (kinetic energy)(field energy)/m^2. The Hamiltonian (3.2) is the tangent vector field equivalent of (2.20), except for the absence of the mass term in the latter and for a factor of two in the divergence of $\mathbf{E}$ term. The three terms in the parenthesis in (3.2) and the five terms outside it add to eight. Terms 1, 2, 3, 4, 5,6 and 7 of (2.20) correspond to terms 2, 5, 4, 3, 7, 8 and 6 of (3.2). The reason for the absence of the m term has already being explained. What is surprising tis that all the other terms are the same except for the factor of two to which we have just made reference

We follow the presentation by Bjorken and Drell. This is a Hamiltonian that displays the different interaction terms between the electron and an applied field in an easily interpretable form. They do not give the form of the equation or on what "function" (actually a two component spinor) this operator is acting upon. We shall see further below why this remark is important.

They apply a unitary transformation three times. The purpose in doing so is to remove all from the equation all operators such as α which couple the large to the small components. It is implicit in this that the equation where (3.2) would be the Hamiltonian is an equation for the positive energy state. More on this below. They describe a Foldy-Wouthuysen transformation as "a canonical transformation which decouples the Dirac equation into two two-component equations; one reduces to the Pauli description in the nonrelativistic limit; the other describes the negative energy states."

In speaking of negative energies, they state that they "and four-component wave functions are the price we must pay in order to have a factorization of H' in (4.1) into a linear Dirac equation." H' is the operator

$$H' = \beta\sqrt{m^2 + p^2}. \tag{3.3}$$

It is worth noticing that the Kähler equation was born linear, and that negative energy solutions need not accompany the positive ones. There is not an ab initio need to separate them if we start with a spinor in the right ideal, and small components, when dealing with the electron, do not then belong to the positron.

We now cite on the Foldy-Wouthuysen transformation from the book "Relativistic Quantum Mechanics and Introduction to Field Theory" by F. J. Yndurain. Let us start with "... of the four solutions of the Dirac equation (with or without a potential) only two are physical". Make what you want of such a statement. You would find in every book on the subject some statement who makes you raise your eyebrows, though any such statement does not raise the same eyebrows.

Yndurain performs two Foldy-Wouthuysen transformations on "the Dirac for a particle in a potential, the sum of a Minkowski vector ($\mathbf{A}$, A_0), a Minkowski fourth component, V_0, and a scalar one, βV_S :

$$H = mc^2\beta + V + V_S\beta + c\alpha\cdot\left(\mathbf{P} - \frac{e}{c}\mathbf{A}\right); \tag{3.4}$$

we have here defined $V \equiv A_0 + V_0$". Here $\mathbf{P}$ is the usual $\mathbf{p}$. His result for "the Foldy-Wouthuysen Hamiltonian" is

$$\begin{aligned} H_{FW} = mc^2\beta + V + V_S\beta + \beta\frac{(\mathbf{P} - e\mathbf{A})^2}{2m} - e\frac{1}{2m}\beta\sigma\cdot\mathbf{B} - \beta\frac{\mathbf{P}^4}{8m^3c^2} \\ - \frac{1}{8mc^2}\left[\alpha\mathbf{P},[\alpha\mathbf{P},V]\right] - \frac{1}{8mc^2}\beta\left\{\alpha\mathbf{P},\{\alpha\mathbf{P},V_s\}\right\}. \end{aligned} \tag{3.5}$$

The author has performed two Foldy-Wouthuysen transformations. Acting on operators (call them h) to the left of the spinor, each Foldy-Wouthuysen transformation yields take $e^T h e^{-T}$. Under the two transformations, the positive energy solutions of the Dirac equation are represented by wave functions

$$\psi_{FW} = e^{T'}e^T\psi + O(c^{-3}). \tag{3.6}$$

($O(c^{-3})$ as in the original). We were interested in this comment, which certainly we are not criticizing. He is certainly treating Foldy-Wouthuysen transformations as gauge transformations. $H\psi$ will go into

$$(e^{T'}e^T H e^{-T}e^{-T'})(e^{T'}e^T\psi) = e^{T'}e^T H\psi. \tag{3.7}$$

Hence, the Dirac equation written as $i\hbar\partial\psi/\partial t = H\psi$ goes to

$$H_{FW}\psi_{FW} = i\hbar e^{T'}e^T\frac{\partial\psi}{\partial t} = i\hbar\frac{\partial\psi_{FW}}{\partial t}. \tag{3.8}$$

Let us finally consider one more approach to the Hamiltonian in a post-Pauli approximation. In the first part of their volume (IV) on Relativistic Quantum mechanics, Landau and Lifchitz reportedly reproduce the treatment of A. Akhieser an V. Berestetski to reach the next approximation in the development of (3.1), They say that they assume that there is only an electric field. They set $\mathbf{A} = \mathbf{0}$ and obtain

$$H = \frac{\mathbf{p}^2}{2m} + e\Phi - \frac{\mathbf{p}^4}{8m^3} \ - \frac{e}{4m^2}\sigma \cdot (\mathbf{E} \times \mathbf{p}) - \frac{e}{8m^2} div\mathbf{E}. \tag{3.9}$$

Again, the method followed causes the absence of the m term. The absence of the m term, as was the case with (2.20) has to do with the fact that they intended to obtain the next approximation for a Schrödinger type equation.

Also absent in (3.9) are the terms 5 and 6 of (3.2). The absence of term 5 is not surprising given that they have assumed $\mathbf{A} = \mathbf{0}$. But the absence of term 6 might be attributed to the fact that making $\mathbf{A} = \mathbf{0}$ makes the electric field incomplete. There is no discrepancy between the $div\mathbf{E}$ term in (3.2) and (3.9). We do not know where the discrepancy by a factor of two with a similar term in (2.20) is born.

3.2 Research suggestions for involvement with the Kähler calculus

Without the need for further comparisons, it is fair to say that these developments of the Dirac equation are not very transparent. Our development of the Kähler equation is, but it remains without explanation why we should be satisfied with an equation for φ rather than an equation for u. It would be desirable to approach the problem without the early separation of the equation for u into φ and χ I suggest readers have here an entry point for research on relativistic quantum mechanics with the Kähler calculus. I suggest the following two problems. I have not even started to do what I am suggesting for fear of getting too involved with them at a "time when I do not have time". So, I do not know whether one gets something of interest and/or easy to handle. Much will depend on the ability of those who may try.

Research problem. Since the β of equation (1.11) is a $0-$form, we can solve for ηp. This can be replaced in the equation obtained by applying η to (1.11). At this point, it will be useful to see what this equation says after replacing α the β for electromagnetic coupling read from (2.10). It seems clear that this will not be directly comparable to (2.20), (3.2) and (3.3) since it will involve the potential rather than the electric and magnetic fields. One can then proceed with different types of approximations only limited by your imagination.

Research problem. One can then try to follow the same process now with (1.19), where the chirality or handedness of the state differential form is defined.

4 Breakthrough energy, Kähler calculus and Schwinger's source theory

Dirac's theory carries a baggage that Kähler's does not. The former has been loaded since birth with the concept of spinor as fundamental, rather than emergent (Notice that u does not even represent a particle in Eq. (1.8), and a need not contain the mass). We want to extract clean energy from wherever. matter holds the greatest density of energy. But, at the nuclear level, energy is not sufficiently well understood. What we have seen in previous sections should be enough to make the point that Kähler's theory holds promise in this regard because it is more powerful, more structured and more comprehensive. Its connection with nuclear physics may be in source theory.

4.1 Schwinger's source theory

The modern expansion of Dirac's theory, with concepts like spin connection, comes at a stiff price and does not even have the right generality. In the first of the already posted chapters, one sees that the Kähler derivative of vector-valued differential forms also depends on the connection on the manifold. Unlike the spinor connection, it will not generally involve Christoffel symbols, either directly or exclusively, unless the affine connection of the manifold is the Levi-Civita connection. To make matters worse, dealing with the generalization of the Dirac operator requires years of specialization by those who eventually work with it. And what has been achieved with it?

As in Dirac's, the concept of mass is not yet emergent in Kähler's theory, but the concept of mechanical and generalized momentum is. Concepts like Foldy-Wouthuysen transformations are unnecessary and cloud the issues. Also in the Kähler calculus, the treatment of the energy operator (we leave the imaginary unit factor for the last day of the summer school) and of Hamiltonians is just a matter of computing with differential equations without resort to operator theory. And most significant is the fact that Kähler's treatment of angular momentum is just a matter of Killing symmetry and properly handling sums of partial derivatives, rather than replacing operators for classical particle magnitudes. Our treatment of relativistic quantum mechanics without operator theory should have made plausible the idea that cutting edge quantum physics need not be quantum field theory, which is operator

based. Nor is the by now forgotten S-matrix theory a suitable alternative. We anticipate that a third option, Schwinger's source theory, constitutes a proxy for what the Kähler calculus will become when used to address the same issues. But it has received far less attention than it deserves.

Source theory is difficult to define. Its major attractiveness is that "the results of quantum electrodynamics are reproduced without the irrelevance of divergences, or renormalizations". It has spacetime emphasis like quantum field theory, but it differs from it in that it is not operator based. Like S-matrix theory, it also has phenomenological emphasis, which we do not view with enthusiasm. But the phenomenology might look less so when approached with the more formal perspective that the Kähler calculus provides.

Schwinger points out that "... in general, particles must be **created** in order to study them, since most of them are unstable. In a general sense this is also true of high-energy stable particles, which must be created in that situation by some device, i.e. an accelerator. One can regard all such creation acts as collisions, in which the necessary properties are transferred from other particles to the one of interest... The other particles in the collision appear only to supply these attributes. They are, in an **abstract** sense, the **source** of the particle in question... We try to represent this abstraction of realistic processes numerically...".And further down, he writes: "Unstable particles eventually decay and the decay process is a detection device. More generally, any detection device can be regarded as removing or **annihilating** the particle. Thus the **source concept** can again be used as an **abstraction** ... with the source acting negatively, as a sink." Bold face has been added.

At a much simpler level, consider the wealth of results we obtained from writing down $\partial u = au$, which is like $y' = f(x)y$ but for a calculus based on Clifford algebra. The formalism leading to it did was not based on physical considerations except for geometry of the spacetime manifold. The Kähler equation does not have physical history. Certainly it borrows from history, but it is one in which it did not take part. It thus acquires physical ascendancy only when we claim that it represents reality. If it could talk, it would advertise itself as follows "If I am given the sum of a constant scalar and a scalar-valued differential $1-$form, I return to you these solutions ...". It does not have its roots in classical physics, which usually takes place through operators.

4.2 Breakthrough energy

The types of Hamiltonians used in nuclear reactions appear not to be clean cut. A translation into a Kähler calculus description of source

theory might could go a long way towards enrichment of the former, and towards a systematic and clean study of Hamiltonians for nuclear reactions. So far, this young but languishing calculus has not yet reached the necessary degree of development.

A case in point of this fussiness is precisely the reaction of deuterium with hydrogen to yield ^{3}He plus heat in a hydrolysis process that uses as cathode a palladium lattice (see next paragraph). After Schwinger argued that the interaction with phonons in the palladium lattice enhanced the p-d fusion rate, O. H. Crawford counterargued that this mechanism does not enhance it (Fusion Technology, Vol 21, March 1992, pp 161-162). This is a beautiful example of a controversy about Hamiltonians for nuclear reactions that, by its very nature, an extended Kähler calculus might help resolve. The core issue is if and how the palladium lattice can diminish the width of the Coulomb barrier so that the reaction could take place in significant amounts. It is precisely in connection with such controversies that an extended Kähler calculus could play a crucial role.

Let us focus on the experimental evidence. Silver, Dash and Keefe (Fusion Technology, Vol 24, Dec. 1993, pp 423-430) examined "with the scanning electron microscope, scanning tunnelling microscope, and atomic force microscope" unusual surface characteristics of an electrolyzed palladium cathode but not on palladium that had not been electrolyzed. More specifically, the unusual features happened when the electrolyte contained hydrogen and deuterium atoms, but not, for instance, when it was constituted by heavy water and sulfuric acid.

There is another important point, which I serendipitously learned about. In the mid to late eighties, physicist Edbertho Leal-Quiros was doing experimental plasma physics with a machine that he himself had helped build as Ph. D. student in the group of Professor M. Prelas at the University of Missouri in Columbia, and that he used for his doctoral thesis. He was creative at building gadgets to measure properties of plasmas given the experience he had acquired at doing so while obtaining his master's degree at the National University of Colombia in Bogota under a German program to help develop physics in that country. He once proudly told me that at some point he was able to measure more plasma parameters than anybody else, and that he had put that knowledge to work when helping build that machine.

Years later, when the program in Columbia was terminated, its university donated the machine to the Metropolitan University of Puerto Rico, where Leal had become a professor. Many years ago he told me that while working with this machine, he used to find from time to time a gamma ray of about the expected energy of about the 5.5 MeVs to be expected if such a reaction took place. Leal, who now holds with his son

David at least a patent on the production of the rare and expensive element ^{3}He, did not have a lattice. He rather had an erratic high energy photon, but not an explanation of where it came from. Since he was working with plasmas, it should not be surprising that the formidable Coulomb barrier might have been jumped from time to time. More recently, they claim to have produced ^{3}He yielding phonons on a crystal, as predicted by Schwinger, but not in palladium crystal.

Finally, there is a very intimate relation between Einstein's failed attempt at unification with teleparallelism —this need not have been so much of a failure— and the Kähler calculus. Hint: Finslerian connections on pseudo-Riemannian metrics show preference for Kähler's over Dirac's theory by virtue of where the two indices of the electromagnetic field fit in geometric quantites of interest in tangent bundle related geometrization of the physics. The last ideas by Einstein, Schwinger and Kähler may have been their best. They were ignored without regard for their merit. One cannot appreciate what one does not care to understand!