A SYSTEMATIC CONSTRUCTION OF REPRESENTATIONS OF QUATERNIONIC TYPE

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The appearance of quaternions in representation theory is usually by accident, both poorly understood yet thought to be deeply meaningful [5, 6]. Examples are the representations of root systems in 3D and 4D in terms of (pure) quaternions, as well as representations of quaternionic type of the polyhedral and other groups.

I have demystified the former in previous work, showing that 4D root systems are induced from 3D root systems in complete generality; in particular, the 3D root systems can only be realised in terms of pure quaternions when the corresponding reflection group contains the inversion [1, 2, 3]. The emergence of the 4D root systems hinges on the Clifford algebra of 3D, or rather its even subalgebra. The spinors describing rotations in 3D (from even products of the reflection generating root vectors) can be endowed with a well-known 4D Euclidean distance. The axioms of a root system are then easily satisfied: firstly, via the Euclidean metric a 3D spinor group can be treated as a collection of vectors in 4D; secondly, Clifford spinorial methods provide a double cover of rotations such that the 4D collection of vectors contains the negatives of those vectors, and thirdly with respect to the 4D Euclidean distance and using some other properties of spinors, the collection of 4D vectors is closed under reflections amongst themselves. These representations of root systems in terms of quaternions therefore systematically hinge purely on the geometry of 3D and the accident that the even subalgebra, i.e. the spinors, is quaternionic.

Here I discuss systematically the representation theory of the intimately related polyhedral groups [4, 2]. The 8D Clifford algebra of 3D space allows one to easily define various representations: the trivial one, parity, the usual 3×3 rotation matrix representation acting on a 3D vector achieved by sandwiching a vector with the corresponding versor, or the 8×8 representation of the group elements as reshuffling the multivector components in the whole 8D algebra under multivector multiplication. The representations we will focus on, however, are those defined by acting with any spinor on another general spinor. This reshuffles the components of the general spinor, which can also be expressed as a 4×4 matrix acting on the spinor in column format. It is not surprising that again because of the quaternionic nature of the even subalgebra the representations of quaternionic type of the polyhedral groups arise naturally and geometrically in a systematic way. Both observations therefore demystify quaternionic phenomena as consequences of 3D geometry, and in particular the 'mysterious deep significance' is simply provided by their spinorial nature – both simple yet underappreciated.

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