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New foundations for geometric algebra[1]

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Abstract. New foundations for geometric algebra are proposed based upon the existing isomorphisms between geometric and matrix algebras. Each geometric algebra always has a faithful real matrix representation with a periodicity of 8. On the other hand, each matrix algebra is always embedded in a geometric algebra of a convenient dimension. The geometric product is also isomorphic to the matrix product, and many vector transformations such as rotations, axial symmetries and Lorentz transformations can be written in a form isomorphic to a similarity transformation of matrices. We collect the idea Dirac applied to develop the relativistic electron equation when he took a basis of matrices for the geometric algebra instead of a basis of geometric vectors. Of course, this way of understanding the geometric algebra requires new definitions: the geometric vector space is defined as the algebraic subspace that generates the rest of the matrix algebra by addition and multiplication; isometries are simply defined as the similarity transformations of matrices as shown above, and finally the norm of any element of the geometric algebra is defined as the n^{th} root of the determinant of its representative matrix of order n. The main idea of this proposal is an arithmetic point of view consisting of reversing the roles of matrix and geometric algebras in the sense that geometric algebra is a way of accessing, working and understanding the most fundamental conception of matrix algebra as the algebra of transformations of multiple quantities.

1 Introduction

In his memoir *On multiple algebra* [1], Josiah Willard Gibbs explored the algebras proposed by several authors in the XIX century in order to multiply multiple quantities (vectors), and he reviewed Grassmann's extension theory, Hamilton's quaternions and Cayley's matrices among others as well as the relations between them. Many kinds of products of vectors have been proposed since then, including Gibbs' skew product of vectors in the Euclidean three-dimensional space [2: 21] (from now on *room space* in order of brevity). What caught my attention was the following phrase of Gibbs [3: 179]:

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> "We have, for example, the tensor of the quaternion[2], which has the important property represented by the equation: $\mathrm{T}(q\,r) = \mathrm{T}q\;\mathrm{T}r$.
>
> There is a scalar quantity related to the linear vector operator which I have represented by the notation $|\Phi|$ and called the *determinant* of Φ. It is in fact the determinant of the matrix by which Φ may be represented, just as the square of the tensor of q (sometimes called the *norm*[3] of q) is the determinant of the matrix by which q is represented. It may also be defined as the product of the latent roots[4] of Φ, just as the square of the tensor of q might be defined as the product of the latent roots of q. Again, it has the property represented by the equation $|\Phi.\Psi| = |\Phi|\,|\psi|$ which corresponds exactly with the preceding equation with both sides squared."

That is, he pointed out that the relation between the determinant of the matrix representation of a quaternion and its norm was a power. Gibbs said that the determinant was the square, but it is the 4th power of the present norm for the regular 4×4 matrix representation:

$$q = a\,i + b\,j + c\,k + d \qquad \Rightarrow \qquad \det q = |q|^4 = \left(a^2 + b^2 + c^2 + d^2\right)^2 \tag{1}$$

I wish to quote another phrase of Gibbs [4, p. 157]:

> "The quaternion affords a convenient notation for rotations. The notation $q\;(\;)\;q^{-1}$, where q is a quaternion and the operand is to be written in the parenthesis, produces on all possible vectors just such changes as a (finite) rotation of a solid body."

That is, if q is represented by a matrix, a rotation is a similarity transformation. In fact, many vector transformations such as rotations, axial symmetries and Lorentz transformations can be written in the form $v' = q^{-1} v\, q$ [5, 6, 7: 27, 8: 19], which is isomorphic to a similarity transformation of matrices. It can be applied not only to vectors, but also to the other elements of geometric algebra.

While searching for a square root of the Klein-Fock equation in order to find the relativistic electron equation, Paul Adrien Maurice Dirac [9] surprisingly took a basis of complex matrices for the space-time geometric algebra instead of taking geometric elements (vectors) as the fundamental entities. Later on, Ettore Majorana [10] found a real 4×4 matrix representation[5] equivalent to Dirac's matrices. The isomorphism between geometric algebras and matrix algebras is well known. Each geometric algebra always has a faithful real matrix representation with a periodicity of 8 [11]:

$$Cl_{p,\,q+8} \cong Cl_{p+8,\,q} \cong Cl_{p,q} \otimes M_{16\times 16}(\mathbb{R}) \tag{2}$$

---

[2] William Rowan Hamilton called *tensor* what we take as the norm nowadays (See *Elements of Quaternions*, **I**: 163).

[3] Hamilton called *norm* the square of our norm, that is, the sum of the squares of the components of a quaternion.

[4] *Latent roots* means *eigenvalues*.

[5] It is curious that the smaller faithful representation of the non-physical Euclidean four-dimensional geometric algebra $Cl_{4,0}$ is included in the complex matrices $M_{4\times 4}(\mathbb{C})$ or, by expansion, in the real $M_{8\times 8}(\mathbb{R})$.

On the other hand, each square matrix algebra is embedded in a geometric algebra of a convenient dimension, while the geometric product is isomorphic to the matrix product. For instance, the algebra of square real 2×2 matrices, $M_{2\times 2}(\mathbb{R})$, is isomorphic to the geometric algebra of the Euclidean plane $Cl_{2,0}$ and also to the geometric algebra of the hyperbolic plane $Cl_{1,1}$ in virtue of the general isomorphism [12]:

$$Cl_{p,q} \cong Cl_{q+1,\, p-1} \tag{3}$$

Another example is Majorana's representation $M_{4\times 4}(\mathbb{R})$, which is a real representation of the space-time geometric algebra $Cl_{3,1}$.

Since all Clifford algebras are included in matrix algebras, I wondered whether the most fundamental concept was matrices or geometric vectors, and if an arithmetic point of view could give us advantage over the geometric point of view with which geometric algebras have been studied until now.

2 Geometric algebra *ab initio*

Leopold Kronecker stated [13]:

"God made the integers, and all the rest is the work of man."

I do not wish to be as radical as him[6] but let us suppose for a moment that the multiple quantities of real numbers are the only tangible reality. Let us search for a rule of multiplication of these multiple quantities taking Gibbs' point of view and without any presupposition about this rule, although we expect to have two algebraic properties: the distributive property and the associative property. The first one is always required for any kind of vector multiplication. The second one is not always required, like in the case of the skew (cross) product, but its presence has clear advantages, especially for algebraic manipulations and geometric equation solving [14]. The most elemental outlining of the transformations of multiple quantities leads us to matrices. If $\mathbf{v} = (v_1 \cdots v_n)$ is a multiple quantity with real components, then we can find any other one $\mathbf{v}' = (v_1' \cdots v_n')$ through a linear transformation represented by a matrix $\mathbf{M} = (m_{ij})$:

$$\begin{pmatrix} v_1' \\ \vdots \\ v_n' \end{pmatrix} = \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \qquad \mathbf{v}' = \mathbf{M}\,\mathbf{v} \tag{4}$$

The distinction between operator (matrix) and operand (multiple quantity) is fictitious since any operand is also an operator. So, the multiple quantity is also an operator and also has a matrix representation a column of which is the column here shown. Note that I am talking about "multiple quantities" instead of "vectors" because the word "vector" needs a more precise definition and I wish to avoid confusion between algebraic vectors (elements of a vectorial space) and geometric vectors (generators of the Clifford algebra). The composition of two linear transformations $\mathbf{M} = (m_{ij})$ and $\mathbf{N} = (n_{ij})$ naturally leads us to the matrix product:

[6] Perhaps if the development of quantum gravity destroys the fiction of the continuity of the room space we shall then agree with Kronecker.

$$\begin{pmatrix} v_1'' \\ \vdots \\ v_n'' \end{pmatrix} = \begin{pmatrix} n_{11} & \cdots & n_{1n} \\ \vdots & \ddots & \vdots \\ n_{n1} & \cdots & n_{nn} \end{pmatrix} \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \tag{5}$$

That is:

$$\mathbf{v''} = \mathbf{P}\ \mathbf{v} \qquad \text{with} \qquad \mathbf{P} = \mathbf{N}\ \mathbf{M} \tag{6}$$

and the multiplication rule:

$$p_{ij} = \sum_k n_{ik} m_{kj} \tag{7}$$

Following a similar way, William Rowan Hamilton discovered quaternions as the operators q which transform geometric vectors in the room space:

$$v' = q\ v \tag{8}$$

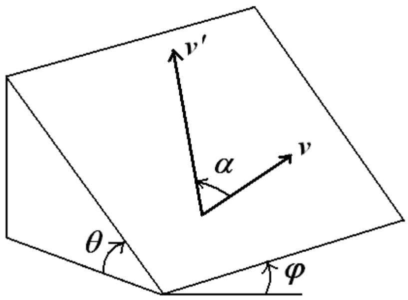

Fig. 1. Quaternion operating upon a vector.

and the rules of their product [15]. He was surprised by the fact that the transformation of three-dimensional vectors required four real quantities, a quaternion, instead of three quantities, which are the inclination θ of the plane, the declination φ, the angle α between both vectors and the ratio of their lengths $|v'|/|v|$ (fig. 1).

Once square matrices, which already contain vectors, have been stated as the fundamental concept of geometric algebra, new definitions must be given in order to work with them.

3 New definitions in geometric algebra

The necessary new definitions that I propose are the following:

1) A *complete geometric algebra* is a square matrix algebra $M_{2^k \times 2^k}(\mathbb{R})$, $k \in \mathbf{N}$. Many geometric algebras are not complete (such as quaternions or $Cl_{4,0}$) because their smallest faithful representation is a subalgebra of a matrix algebra of the same order. The space-time geometric algebra is a complete geometric algebra because $Cl_{3,1} \cong M_{4\times 4}(\mathbb{R})$.
2) The *generating vector space* (the *geometric vector space*) is the set of matrices and their linear combinations (a vectorial subspace) that generate the whole geometric algebra by multiplication. The concept is similar to the set of generators of a discrete group, but applied to a continuous group. The elements of the generating vector space are the *geometric vectors*.
3) The *norm of every element* of a geometric algebra $M_{n\times n}(\mathbb{R})$ is the n^{th} root of the determinant of its representative matrix:

$$|\mathbf{M}_{n\times n}| = \sqrt[n]{\det \mathbf{M}} \tag{9}$$

For instance, the subalgebra of quaternions is given by:

$$a+b\,i+c\,j+d\,k=\begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \tag{10}$$

whose norm is obtained from the 4[th] root of the matrix determinant:

$$\left|a+b\,i+c\,j+d\,k\right|=\sqrt[4]{\det\begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}}=\sqrt{a^2+b^2+c^2+d^2} \tag{11}$$

The norm can be a real number, an imaginary number and also zero since all the complete geometric algebras have divisors of zero. According to Frobenius' theorem [16], the only division associative algebras[7] are the real numbers, the complex numbers and quaternions.

4) *Isometries* are defined as the similarity transformations of matrices:

$$\mathbf{M'}=\mathbf{P}^{-1}\mathbf{M}\,\mathbf{P} \quad \text{with} \quad \det\mathbf{P}\neq 0 \qquad \Rightarrow \qquad \det\mathbf{M'}=\det\mathbf{M} \tag{12}$$

because they preserve the determinant and hence the norm.

5) Two elements are said to be *equivalent* (the equivalence will be represented by ~) if their matrices can be transformed one into the other through an isometry, that is, through a similarity transformation. To have the same norm and determinant does not imply to be equivalent since similar matrices have the same eigenvalues and the determinant is only their product. For instance, in the space-time algebra $Cl_{3,1}\cong M_{4\times 4}(\mathbb{R})$, we have $e_1 \sim e_2 \sim e_3$ but they are not equivalent to e_0 although $\det e_1=\det e_2=\det e_3=\det e_0=1$.

6) A *unity* is a matrix whose square power is equal to $\pm\mathbf{I}$, and whose determinant is equal to 1 (from order 4 on). The unities can be found through the tensor product of the four unities of $M_{2\times 2}(\mathbb{R})$, the smallest complete geometric algebra:

$$1=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad e_1=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e_2=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad e_{12}=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{13}$$

For instance, a unity of $M_{4\times 4}(\mathbb{R})$ is:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\otimes\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{14}$$

Of course, any matrix similar to this one is also a unity.

7) Two elements are said to be *perpendicular* if they anticommute.

[7] Algebras without divisors of zero.

4 Consequences of the new definitions

4.1 The Pythagorean and pseudo-Pythagorean theorems

Any set of perpendicular unities fulfils the Pythagorean or pseudo-Pythagorean theorem. Let $\{\mathbf{E}_i\}$ and $\mathbf{M}$ be respectively a set of perpendicular unities and a linear combination of them:

$$\mathbf{E}_i \in M_{n\times n}(\mathbb{R}) \qquad \mathbf{E}_i^2 = \chi_i \mathbf{I} \qquad \chi_i = \pm 1 \qquad \forall i \neq j \quad \mathbf{E}_i\mathbf{E}_j = -\mathbf{E}_j\mathbf{E}_i \tag{15}$$

$$\mathbf{M} = \sum_i \alpha_i \mathbf{E}_i \quad \Rightarrow \quad \mathbf{M}^2 = \left(\sum_i \alpha_i \mathbf{E}_i\right)^2 = \sum_i \alpha_i^2 \mathbf{E}_i^2 = \mathbf{I} \sum_i \alpha_i^2 \chi_i \tag{16}$$

Then:

$$\det \mathbf{M}^2 = \det\left(\mathbf{I} \sum_i \alpha_i^2 \chi_i\right) = \left(\sum_i \alpha_i^2 \chi_i\right)^n \quad \Rightarrow \quad \det \mathbf{M} = \pm\left(\sum_i \alpha_i^2 \chi_i\right)^{n/2} \tag{17}$$

$$|\mathbf{M}| = \sqrt[n]{\det \mathbf{M}} = \sqrt{\pm \sum_i \alpha_i^2 \chi_i} \tag{18}$$

For instance, the determinant of a bivector of the space-time geometric algebra $Cl_{3,1}$ does not fulfil the Pythagorean theorem:

$$\det(a\, e_{01} + b\, e_{02} + c\, e_{03} + f\, e_{23} + g\, e_{31} + h\, e_{12}) = \\ \left(a^2 + b^2 + c^2 - f^2 - g^2 - h^2\right)^2 + 4(a\, f + b\, g + c\, h)^2 \tag{19}$$

because $e_{01}e_{23} = e_{23}e_{01}$ and so on. However, if the first or the second triple of components vanishes, the norm is then given by the Pythagorean theorem:

$$|\, a\, e_{01} + b\, e_{02} + c\, e_{03}| = \sqrt{a^2 + b^2 + c^2} \qquad |\, f\, e_{23} + g\, e_{31} + h\, e_{12}| = \sqrt{f^2 + g^2 + h^2} \tag{20}$$

because the remaining unit bivectors are perpendicular.

4.2 Generality of the expression of isometries as similarity transformation

The expression of isometries as similarity transformations is general and can be applied to any element of the geometric algebra. Let us suppose for a moment that this expression can only be applied to geometric vectors. Then, it can be applied to geometric products of vectors:

$$v' = q^{-1} v\, q \quad \Rightarrow \quad v_1' v_2' = q^{-1} v_1 q\, q^{-1} v_2 q = q^{-1} v_1 v_2 q \tag{21}$$

and also to exterior products of vectors and their linear combinations, that is, to any element of second degree:

$$(v_1 \wedge v_2)' = v_1' \wedge v_2' = \frac{1}{2}(v_1' v_2' - v_2' v_1') = \frac{1}{2} q^{-1} (v_1 v_2 - v_2 v_1)\, q = q^{-1}\, v_1 \wedge v_2\, q \tag{22}$$

and so on for any degree, that is, for any element of the geometric algebra. Nowadays, certain isometry operators are written in a form that is only valid for geometric vectors but not for other elements of the geometric algebra. For instance, a rotation of angle θ of a vector in the Euclidean plane can be written in $Cl_{2,0}$ as [17: 52]:

$$v' = v\left(\cos\theta + e_{12}\sin\theta\right) \qquad v = v_1 e_1 + v_2 e_2 \tag{23}$$

but the application of this operator to a complex number $a + b\,e_{12}$ changes its argument. However, complex numbers are geometric products (or quotients) of two plane vectors. Both vectors are turned through the same angle of rotation θ, so that the angle α between both vectors does not change, and therefore complex numbers must be preserved [7: 27] (fig. 2). We can only obtain this result by means of the half-angle operator:

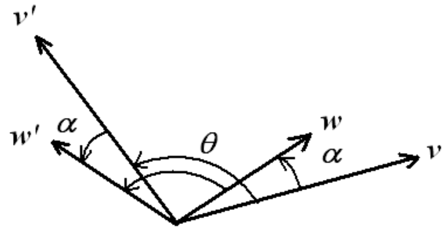

Fig. 2. Preservation, upon a rotation, of the angle between two plane vectors and their lengths, and therefore of their product or quotient, a complex number.

$$v' = \left(\cos\frac{\theta}{2} - e_{12}\sin\frac{\theta}{2}\right) v \left(\cos\frac{\theta}{2} + e_{12}\sin\frac{\theta}{2}\right) \tag{24}$$

which is a similarity transformation. Now complex numbers are preserved because of their commutative property:

$$z' = \left(\cos\frac{\theta}{2} - e_{12}\sin\frac{\theta}{2}\right) z \left(\cos\frac{\theta}{2} + e_{12}\sin\frac{\theta}{2}\right) = z \qquad z = a + b\,e_{12} \tag{25}$$

4.3 Isometries of perpendicular geometric vectors

Isometries transform perpendicular geometric vectors into perpendicular geometric vectors, which can be easily proven:

$$\mathbf{E}_i\mathbf{E}_j = -\mathbf{E}_j\mathbf{E}_i \quad \Rightarrow \quad \mathbf{P}^{-1}\mathbf{E}_i\mathbf{P}\,\mathbf{P}^{-1}\mathbf{E}_j\mathbf{P} = -\mathbf{P}^{-1}\mathbf{E}_j\mathbf{P}\,\mathbf{P}^{-1}\mathbf{E}_i\mathbf{P}$$
$$\Rightarrow \quad \mathbf{E}_i'\mathbf{E}_j' = -\mathbf{E}_j'\mathbf{E}_i' \tag{26}$$

because $\mathbf{P}\,\mathbf{P}^{-1} = \mathbf{I}$. Both vectors can lie either in an Euclidean plane or in a hyperbolic plane. In the second case, two vectors are perpendicular if we "see" their directions as being symmetric with respect to the quadrant bisectors [7: 156]. Fig. 3 shows how an isometry, such as a Lorentz transformation, transforms a pair of perpendicular vectors u, v into another pair of perpendicular vectors u', v'.

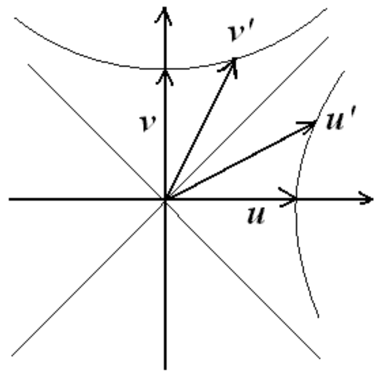

Fig. 3. Transformation of two perpendicular vectors *u*, *v* into another pair of perpendicular vectors *u'*, *v'* under an isometry in a hyperbolic plane.

4.4 Perpendicular vectors as generators of the geometric algebra

Any product of a number of perpendicular unities less than or equal to the dimension of the generating space is linearly independent of them and of other products of lower degree. It follows immediately from the identity between the geometric product of perpendicular vectors and their exterior product:

$$\forall i \neq j \qquad \mathbf{E}_i\mathbf{E}_j = -\mathbf{E}_j\mathbf{E}_i \qquad \Rightarrow \qquad \mathbf{E}_i \cdots \mathbf{E}_k = \mathbf{E}_i \wedge \cdots \wedge \mathbf{E}_k \qquad i < \cdots < k \tag{27}$$

because each exterior product is a multiplication by a component perpendicular to the subspace generated by the previous vectors. This is true up to the dimension of the generating space. We can also prove this linear independence in another way. For instance, the complete geometric algebra $M_{2\times2}(\mathbb{R})$ has two perpendicular generating unities $\mathbf{E}_1$ and $\mathbf{E}_2$:

$$\mathbf{E}_1\mathbf{E}_2 = -\mathbf{E}_2\mathbf{E}_1 \qquad \mathbf{E}_i^2 = \chi_i = \pm 1 \tag{28}$$

Let us suppose that their product is a linear combination of the generating unities and the identity:

$$\mathbf{E}_1\mathbf{E}_2 = \alpha_0\mathbf{I} + \alpha_1\mathbf{E}_1 + \alpha_2\mathbf{E}_2 \tag{29}$$

If we multiply the equality by $\mathbf{E}_1$ both on the left and on the right we obtain:

$$\left.\begin{array}{r} \mathbf{E}_1^2\mathbf{E}_2 = \chi_1\mathbf{E}_2 = \alpha_0\mathbf{E}_1 + \alpha_1\chi_1 + \alpha_2\mathbf{E}_{12} \\ \mathbf{E}_1\mathbf{E}_2\mathbf{E}_1 = -\chi_1\mathbf{E}_2 = \alpha_0\mathbf{E}_1 + \alpha_1\chi_1 - \alpha_2\mathbf{E}_{12} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \alpha_0 = \alpha_1 = 0 \\ \alpha_2^2 = \chi_1 \end{array}\right. \tag{30}$$

If we multiply the equality by $\mathbf{E}_2$ both on the left and on the right we obtain:

$$\left.\begin{array}{r} \mathbf{E}_2\mathbf{E}_1\mathbf{E}_2 = -\chi_2\mathbf{E}_1 = \alpha_0\mathbf{E}_2 - \alpha_1\mathbf{E}_{12} + \alpha_2\chi_2 \\ \mathbf{E}_1\mathbf{E}_2^2 = \chi_2\mathbf{E}_1 = \alpha_0\mathbf{E}_2 + \alpha_1\mathbf{E}_{12} + \alpha_2\chi_2 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \alpha_0 = \alpha_2 = 0 \\ \alpha_1^2 = \chi_2 \end{array}\right. \tag{31}$$

a result which comes into contradiction with the former result. Therefore, this proves that our hypothesis that $\mathbf{E}_1\mathbf{E}_2$ is a linear combination of $\{\mathbf{I}, \mathbf{E}_1, \mathbf{E}_2\}$ is false, whence it follows that the set $\{\mathbf{I}, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_1\mathbf{E}_2\}$ is a basis of $M_{2\times2}(\mathbb{R})$. A proof based on this line of reasoning which is general for any dimension was already given by Marcel Riesz [18].

4.5 Reflections in the space-time geometric algebra

Reflections need a special mention. When talking with Prof. L. Dorst and Prof. H. Pijls during the ECM 2008 conference in Amsterdam about my supposition that isometries are similarity transformations, they replied that the expression for reflections is not a similarity transformation since [19, 20]:

$$v' = -a^{-1}\, v\, a \tag{32}$$

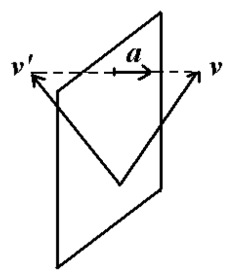

Fig. 4. Reflection of a vector in a plane.

where v is a geometric vector and a is a vector perpendicular to the plane of reflection (fig. 4). The first objection to this expression is the fact that it can only be applied to geometric vectors, but not to other elements of the geometric algebra such as bivectors. The modification which I proposed [8: 36] was to write it as a similarity transformation in the following way:

$$r = e_0 a \qquad \Rightarrow \qquad r^{-1} = -a^{-1} e_0 \tag{33}$$

$$v' = r^{-1} v\, r = -a^{-1} e_0 v\, e_0 a = -a^{-1}\, v\, a \tag{34}$$

where e_0 is the time unitary vector of the space-time geometric algebra $Cl_{3,1}$ and $e_0^2 = -1$. Of course it has a consequence: this operator changes the sign of the time component:

$$e_0{}' = r^{-1} e_0\, r = -a^{-1}\, e_0 e_0\, e_0 a = a^{-1} e_0 a = -a^{-1} a\, e_0 = -e_0 \tag{35}$$

That is, a reflection would be an isometry reversing one spatial direction and also the time direction. We can discuss widely about whether the reversal of one spatial and the temporal components must be linked or not in a reflection. The physical world does not remain invariant under reflections because there are physical processes, driven by weak interactions, whose mirror image has a very much lower probability [21]. However, physical invariance is preserved under the CPT transformation[8] [22], that is, if time is also reversed. On the other hand, the biological world has chosen one side of the mirror: all the proteins of the superior species are built with the L-amino acids while their mirror images, D-amino acids, are absent from the most biological structures. Anyway, we may wonder whether a reflection without time reversal can be a similarity transformation or not. Let us see how a generic element of the space-time geometric algebra $Cl_{3,1}$:

$$\begin{aligned} w = a + b\, e_0 + c\, e_1 + d\, e_2 + e\, e_3 + f\, e_{01} + g\, e_{02} + h\, e_{03} + i\, e_{23} + j\, e_{31} + k\, e_{12} \\ + l\, e_{023} + m\, e_{031} + n\, e_{012} + o\, e_{123} + p\, e_{0123} \end{aligned} \tag{36}$$

changes under a reflection in the plane e_{23}, which produces the reversal $e_1 \to -e_1$:

$$\begin{aligned} w' = a + b\, e_0 - c\, e_1 + d\, e_2 + e\, e_3 - f\, e_{01} + g\, e_{02} + h\, e_{03} + i\, e_{23} - j\, e_{31} - k\, e_{12} \\ + l\, e_{023} - m\, e_{031} - n\, e_{012} - o\, e_{123} - p\, e_{0123} \end{aligned} \tag{37}$$

The characteristic polynomials of both elements[9] are:

$$\det(w - \lambda) = \begin{vmatrix} a+d+h-l-\lambda & b+e-g+i & f+j+n+o & -c+k-m+p \\ -b+e-g-i & a-d-h-l-\lambda & c+k-m-p & f-j-n+o \\ f-j+n-o & c-k-m+p & a+d-h+l-\lambda & -b+e+g+i \\ -c-k-m-p & f+j-n-o & b+e+g-i & a-d+h+l-\lambda \end{vmatrix} \tag{38}$$

$$\det(w' - \lambda) = \begin{vmatrix} a+d+h-l-\lambda & b+e-g+i & -f-j-n-o & c-k+m-p \\ -b+e-g-i & a-d-h-l-\lambda & -c-k+m+p & -f+j+n-o \\ -f+j-n+o & -c+k+m-p & a+d-h+l-\lambda & -b+e+g+i \\ c+k+m+p & -f-j+n+o & b+e+g-i & a-d+h+l-\lambda \end{vmatrix} \tag{39}$$

[8] Charge conjugation, parity or spatial inversion, and time reversal.

[9] I have built these determinants with the matrix basis given in [8: 11]. Notwithstanding this, all the bases of $Cl_{3,1}$ are equivalent and they therefore have the same characteristic polynomial (38), although the matrix elements can change depending on the chosen basis.

In fact, it reduces to a change of sign of all the matrix elements in the highest right square and in the lowest left square. Both determinants are equal, and the characteristic polynomials are identical. Therefore, the existence of a similarity transformation for this reflection cannot be discarded although it is necessary that both matrices have the same invariant factors [23]. When this talk was given (on July 6th in the IKM 2012[10]) I said that this question should be clarified soon. While writing this paper I have found out that this reflection is really a similarity transformation with operator e_{023}:

$$w' = e_{023}^{-1}\, w\, e_{023} \qquad\qquad e_{023}^{-1} = e_{023} \tag{40}$$

A question that immediately follows is whether reflections are similarity transformations in geometric algebras of any dimension or not. The answer is given in 4.7, since first we need the result of 4.6.

4.6 Number of generators of a complete geometric algebra

In a complete geometric algebra $M_{2^k \times 2^k}(\mathbb{R})$ the maximum number of perpendicular unities, leaving the pseudoscalar aside, is $n = 2k$. It is well known that a geometric algebra generated by a geometric space of dimension n has dimension 2^n because:

$$\dim Cl_{p,q} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n \qquad\qquad n = p + q \tag{41}$$

Then, the dimension of this geometric algebra must be equal to the dimension of the linear space of the matrix algebra so that:

$$2^n = 2^k \times 2^k \qquad \Rightarrow \qquad n = 2k \tag{42}$$

The pseudoscalar $e_{1\cdots n}$ always anticommutes with all geometric vectors if the generating space has even dimension:

$$k \in \mathbf{N} \qquad n = 2k \qquad \Rightarrow \qquad e_i e_{1\cdots n} = -e_{1\cdots n} e_i \qquad \forall i \tag{43}$$

When counting the pseudoscalar, the maximum number of perpendicular unities is $2k+1$. For instance, in $M_{4\times 4}(\mathbb{R})$ the maximum number of perpendicular unities is 4 plus the pseudoscalar, while in $M_{8\times 8}(\mathbb{R})$ the maximum number of perpendicular unities is 6 plus the pseudoscalar because $2^6 = 8 \times 8$. However, in virtue of the isomorphisms $Cl_{p,q} \cong Cl_{q+1,\,p-1}$ and $Cl_{p,q} \cong Cl_{p-4,q+4}$ for $p \geq 4$ [12], there are two or more non-equivalent sets of unities generating these geometric algebras [11]:

$$M_{4\times 4}(\mathbb{R}) \cong Cl_{3,1} \cong Cl_{2,2} \tag{44}$$

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[10] IKM 2012, International Conference on the Applications of Computer Science and Mathematics in Architecture and Civil Engineering. Conference held in the historical town of Weimar at the Bauhaus-Universitët from 4th to 6th July 2012.

$$M_{8\times 8}(\mathbb{R}) \cong Cl_{0,6} \cong Cl_{3,3} \cong Cl_{4,2} \tag{45}$$

The isomorphism $Cl_{p,q} \cong Cl_{q+1,\, p-1}$ is displayed in the fact that, in the set of perpendicular unities $\{e_1, \cdots e_n, e_{1\cdots n}\}$, we can take as generating space $\langle e_1, \cdots, e_n \rangle$ or $\langle e_1, \cdots, e_{i-1}, e_{i+1}, e_n, e_{1\cdots n} \rangle$ changing one vector e_i for the pseudoscalar $e_{1\cdots n}$. If both of them have the same square the Clifford structure is the same, but if their squares have different signs the Clifford structure changes according to this isomorphism.

4.7 Reflections in a complete geometric algebra

Reflections in an even geometric algebra $Cl_{p,q}$ ($n = p + q$ even) are similarity transformations. The proof begins when taking into consideration that axial symmetries change the sign of perpendicular components and retain the sign of the proportional component:

$$\forall i \neq j \qquad e_i^{-1} e_j e_i = -e_j \tag{46}$$

$$e_i^{-1} e_i e_i = e_i \tag{47}$$

For instance, if e_1 is the operator, then the isometry is $e_1 \to e_1$, $e_2 \to -e_2$, $e_3 \to -e_3$ and so on. This fact is independent of the sign of the square of all the unities. Now we wish to change this operator for its dual by means of introducing the pseudoscalar $e_{1\cdots n}$. Let us indicate $e_{1\cdots n}^2 = \upsilon = \pm 1$ and let us introduce it in eqn. (46) in order to get the dual operators:

$$e_i^{-1} e_j e_i = \upsilon\, e_i^{-1} e_{1\cdots n}^2 e_j e_i = -\upsilon\, e_i^{-1} e_{1\cdots n} e_j e_{1\cdots n} e_i = -\upsilon \chi_i\, e_i e_{1\cdots n} e_j e_{1\cdots n} e_i \tag{48}$$

where $\chi_i = e_i^2$. Now we cancel the factors e_i with those contained in the pseudoscalar:

$$e_i^{-1} e_j e_i = \upsilon \chi_i e_{1\cdots i-1, i+1 \cdots n} e_j e_{1\cdots i-1, i+1\cdots n} \tag{49}$$

The cancellation of one factor yields $\pm \chi_i$ depending on the parity of the place it occupies in the pseudoscalar, but the cancellation of both factors for *n* even always results in a single negative sign. Therefore from (46) and (49):

$$\upsilon \chi_i e_{1\cdots i-1, i+1\cdots n} e_j e_{1\cdots i-1, i+1\cdots n} = -e_j \tag{50}$$

On the other hand we have:

$$\chi_i e_{1\cdots i-1, i+1\cdots n} e_{1\cdots i-1, i+1\cdots n} = e_{1\cdots i-1, i+1\cdots n} e_i^2 e_{1\cdots i-1, i+1\cdots n} = -e_{1\cdots n}^2 = -\upsilon \tag{51}$$

This means that the inverse of the reflection operator $e_{1\cdots i-1, i+1\cdots n}$ is:

$$e_{1\cdots i-1, i+1\cdots n}^{-1} = -\upsilon \chi_i e_{1\cdots i-1, i+1\cdots n} \tag{52}$$

From (50) and (52) we conclude that:

$$e_{1\cdots i-1,i+1\cdots n}^{-1} e_j e_{1\cdots i-1,i+1\cdots n} = e_j \qquad \forall i \neq j \tag{53}$$

but at the same time we also have:

$$e_{1\cdots i-1,i+1\cdots n}^{-1} e_i e_{1\cdots i-1,i+1\cdots n} = -e_{1\cdots i-1,i+1\cdots n}^{-1} e_{1\cdots i-1,i+1\cdots n} e_i = -e_i \tag{54}$$

Summarizing, the reflection in the hyperplane perpendicular to e_i is a similarity transformation with operator $e_{1\cdots i-1,i+1\cdots n}$. This result does not depend on the sign of the square of the pseudoscalar $e_{1\cdots n}^2 = \upsilon$, and it only needs n to be even. One consequence is the fact that reflections are similarity transformations in all the complete geometric algebras because their generating spaces always have an even dimension.

If n is odd the pseudoscalar $e_{1\cdots n}$ commutes with all vectors:

$$n = 2k+1 \quad k \in \mathbb{N} \qquad e_i e_{1\cdots n} = e_{1\cdots n} e_i \tag{55}$$

Introducing the square of the pseudoscalar in (55) in order to get the dual operators:

$$e_i^{-1} e_j e_i = \upsilon\, e_i^{-1} e_{1\cdots n}^2 e_j e_i = \upsilon\, e_i^{-1} e_{1\cdots n} e_j e_{1\cdots n} e_i = \upsilon \chi_i e_i e_{1\cdots n} e_j e_{1\cdots n} e_i \tag{56}$$

Now we cancel the factors e_i with those contained in the pseudoscalar which yields a positive sign for n odd:

$$e_i^{-1} e_j e_i = \upsilon \chi_i e_{1\cdots i-1,i+1\cdots n} e_j e_{1\cdots i-1,i+1,\cdots n} \tag{57}$$

Therefore:

$$\upsilon\ \chi_i e_{1\cdots i-1,i+1\cdots n} e_j e_{1\cdots i-1,i+1\cdots n} = -e_j \qquad \forall i \neq j \tag{58}$$

On the other hand we have:

$$\chi_i e_{1\cdots i-1,i+1\cdots n} e_{1\cdots i-1,i+1\cdots n} = e_{1\cdots i-1,i+1\cdots n} e_i^2 e_{1\cdots i-1,i+1\cdots n} = e_{1\cdots n}^2 = \upsilon \tag{59}$$

This means that the inverse of the operator $e_{1\cdots i-1,i+1\cdots n}$ is:

$$e_{1\cdots i-1,i+1\cdots n}^{-1} = \upsilon \chi_i e_{1\cdots i-1,i+1\cdots n} \tag{60}$$

From (58) and (60) we conclude that:

$$e_{1\cdots i-1,i+1\cdots n}^{-1} e_j e_{1\cdots i-1,i+1\cdots n} = -e_j \qquad \forall i \neq j \tag{61}$$

but at the same time we also have:

$$e_{1\cdots i-1,i+1\cdots n}^{-1} e_i e_{1\cdots i-1,i+1\cdots n} = e_{1\cdots i-1,i+1\cdots n}^{-1} e_{1\cdots i-1,i+1\cdots n} e_i = e_i \tag{62}$$

which is not a reflection but an axial symmetry (the transformation obtained by means of the operator e_i). Therefore, reflections cannot be written as similarity transformations in odd geometric algebras ($n = p + q$ odd). This result does not depend on the sign of the square of the pseudoscalar $e_{1\cdots n}^2 = \upsilon = \pm 1$. That is the reason why reflections in the room space cannot be written as similarity transformations in the geometric algebra $Cl_{3,0}$, which yields two distinct and non equivalent orientations of the geometric vector basis $\{e_1, e_2, e_3\}$. However, the incompleteness of odd geometric algebras is shown by the fact that there always exist matrices with the order of their lowest faithful representation (although not belonging to them) that allow to write reflections as similarity transformations. This is due to the fact that matrix representations of geometric algebras always have an even order. In the complete geometric algebras, the bases obtained from $\{e_1, \cdots, e_i, \cdots e_n\}$ by reversing one unity $e_i \rightarrow -e_i$ or more unities are all equivalent through reflections:

$$\{e_1, \cdots, e_i, \cdots e_n\} \sim \{e_1, \cdots, -e_i, \cdots e_n\} \tag{63}$$

4.8 Duality

Duality can be a similarity transformation in complete geometric algebras if a suitable Clifford structure is chosen. A *Clifford structure* is a set of generating unities and their products which are a basis of the considered matrix algebra[11]. I already showed [8: 40] that, in the space-time algebra, duality is a similarity transformation. Let us extend this result if possible to higher dimensions. Taking the same duality operator $1 + e_{1\ldots n}$ as that found in the space-time, we have:

$$(1 + e_{1\ldots n})(1 - e_{1\ldots n}) = 1 - e_{1\ldots n}^2 = 1 - \upsilon \tag{64}$$

For $\upsilon = -1$ this product does not vanish and we obtain the inverse of the duality operator:

$$(1 + e_{1\ldots n})^{-1} = \frac{1}{2}(1 - e_{1\ldots n}) \tag{65}$$

Applying a similarity transformation with the duality operator to a unit vector we have:

$$e_i' = (1 + e_{1\ldots n})^{-1} e_i (1 + e_{1\ldots n}) = \frac{1}{2}(1 - e_{1\ldots n}) e_i (1 + e_{1\ldots n}) = \frac{1}{2}(e_i e_{1\ldots n} - e_{1\ldots n} e_i) \tag{66}$$

If *n* is even, by means of (43) we arrive at:

$$e_i' = e_i e_{1\cdots n} \tag{67}$$

[11] A Clifford structure $Cl_{p,q}$ is the same if its generating unities are changed for other equivalent unities. The characteristic of a Clifford structure is its signature p, q, that is, how many generating unities with square $+1$ and how many generating unities with square -1 it has.

which is exactly the dual, although the final sign will depend on the sign of the square $e_i^2 = \chi_i$ and on the parity of the place of e_i in the pseudoscalar. The requirements for this result are *n* even and $e_{1\cdots n}^2 = -1$. When is this last condition fulfilled? Let us see an example:

$$e_{1234}\, e_{1234} = (-1)^3 e_1^2 e_{234} e_{234} = (-1)^{3+2} e_1^2 e_2^2 e_{34} e_{34} = (-1)^{3+2+1} e_1^2 e_2^2 e_3^2 e_4^2 \tag{68}$$

The numbers of swaps for $e_{1\cdots n}^2$ are the triangular number:

$$t_{n-1} = n-1+n-2+\cdots+1 = \frac{n\,(n-1)}{2} \tag{69}$$

and the final result is therefore [24]:

$$\upsilon = e_{1\cdots n}^2 = (-1)^{t_{n-1}} e_1^2 \cdots e_n^2 = (-1)^{\frac{n\,(n-1)}{2}+q} \qquad e_{1\cdots n} \in Cl_{p,q} \tag{70}$$

since there are only *q* squares equal to –1. Now:

$$\upsilon = e_{1\cdots n}^2 = -1 \quad \Rightarrow \quad \frac{n\,(n-1)}{2} + q \quad \text{odd} \tag{71}$$

We have two cases. If $m \in \mathbb{N}$ then:

$$n = 4m \quad \Rightarrow \quad \frac{n\,(n-1)}{2} + q = 2m(4m-1) + q \quad \Rightarrow \quad q \text{ odd} \tag{72}$$

$$n = 4m+2 \quad \Rightarrow \quad \frac{n\,(n-1)}{2} + q = (2m+1)(4m+1) + q \quad \Rightarrow \quad q \text{ even} \tag{73}$$

After a short analysis, both cases can be gathered in the condition:

$$p - q \equiv 2 \pmod 4 \tag{74}$$

The first examples of the Clifford structures that fulfil the former results [25] are given in table 1:

$n = p + q = 2k$	2	4	6	8	10
$M_{2^k \times 2^k}(\mathbb{R})$	$M_{2\times 2}$	$M_{4\times 4}$	$M_{8\times 8}$	$M_{16\times 16}$	$M_{32\times 32}$
$Cl_{p,q}$	$Cl_{2,0}$	$Cl_{3,1}$	$Cl_{4,2}$ $Cl_{0,6}$	$Cl_{5,3}$ $Cl_{1,7}$	$Cl_{10,0}$ $Cl_{6,4}$ $Cl_{2,8}$

Table 1. Clifford structures for complete geometric algebras up to *n*=10 where duality is also a similarity transformation.

In the algebras $Cl_{m,m}$ which are isomorphic to these ones, we cannot write duality as similarity transformations because $e_{1\cdots 2m}^2 = 1$ according to (70). Then, if $e_1^2 = 1$ the square of its dual is $e_{2\cdots 2m}^2 = -1$ or the reverse. For example, in $Cl_{2,2}$ we have:

$$e_1^2 = e_2^2 = 1 \qquad e_3^2 = e_4^2 = -1 \tag{75}$$

The dual of e_1 is e_{234} whose square is:

$$e_{234}^2 = e_{234}e_{234} = e_2^2 e_{34}e_{34} = -e_2^2 e_3^2 e_4^2 = -1 \tag{76}$$

Since e_1 has square $+1$ and e_{234} has square -1, they cannot be equivalent in any way including duality. Therefore duality is not a similarity transformation in $Cl_{2,2}$. This algebra is isomorphic to the space-time algebra:

$$Cl_{2,2} \cong Cl_{3,1} \cong M_{4\times 4}(\mathbb{R}) \tag{77}$$

and both are complete geometric algebras. However, by taking the structure $Cl_{3,1}$ instead of $Cl_{2,2}$ for the algebra of the square matrices of order 4 we gain insight because duality then becomes a similarity transformation.

In the case that n is odd, duality is never a similarity transformation because the duals of the unities e_i, the (n−1)-multivectors $e_{1\cdots i-1,i+1,n}$, contain an even number of vectors, and they cannot generate the whole geometric algebra but only the subalgebra containing elements of even degree. If $\{e_i\}$ generate the whole algebra and $\{e_{1\cdots i-1,i+1,n}\}$ only the subalgebra, both sets cannot be equivalent. We can pass from the first ones to the second ones through duality, but it is not a similarity transformation because matrix similarity is always an equivalence relation. As an example, we can pass in the room space from e_{23} to e_1 through the space duality as Hamilton did, but they are not equivalent because $e_1^2 = 1$ and $e_{23}^2 = -1$. Equivalent unities always have the same square. While the vectors $\{e_1, e_2, e_3\}$ generate the whole $Cl_{3,0}$, the duals $\{e_{23}, e_{31}, e_{12}\}$ only generate the quaternion subalgebra.

More definition improvements, rigorous proofs of those statements here outlined but not proven yet and new refinements must be carried out in future works. The knowledge we have on Clifford algebras will be very helpful in this task.

5 Conclusions

If we take multiple quantities as fundamental entities, then the matrix theory naturally follows from their transformations, and the matrix product from the composition of transformations. In this framework, a geometric algebra is defined as a matrix algebra or subalgebra that is closed under addition and multiplication, and that is generated by the unities obtained from the tensor product of the unities of $M_{2,2}(\mathbb{R})$. A complete geometric algebra is defined as a matrix algebra isomorphic to a geometric algebra over the real numbers, which only happens for $M_{2^k\times 2^k}(\mathbb{R})$. Searching for a generalization of the norm of a complex number or a quaternion, we wish the norm of a product of two elements to be equal to the product of their norms. The unique quantity that fulfils this equality is the determinant, because the determinant of a product of two matrices is equal to the product of their determinants. In order to fit this new norm to the norms of complex numbers, quaternions and vectors, the n^{th} root of the determinant must be taken, where $n = 2^k$ is the order of the square matrix algebra. Since n is always an even number, the norm $|\mathbf{M}|$ of a matrix $\mathbf{M}$ can be a real or an imaginary positive number, which fulfils $|\mathbf{M}\,\mathbf{N}| = \pm|\mathbf{M}|\,|\mathbf{N}|$. This definition of the norm of an element of a geometric algebra is independent of the order of the matrix representation and it fills a void in Clifford algebras theory, since the norm of elements with mixed degree has not been unambiguously defined until now, except for special cases such as quaternions.

On the other hand, an isometry is defined as a matrix similarity transformation, which preserves the determinant and therefore the norm. The advantage of this definition is the fact that the same operator can be applied to any element of geometric algebra. Isometries transform perpendicular vectors into perpendicular vectors. A new definition for unities is also given as matrices with square power equal to $\pm\mathbf{I}$ and determinant equal to 1 (for $n \geq 4$) that are obtained from tensor product of the unities of $M_{2\times2}(\mathbb{R})$. Any matrix equivalent (through a similarity transformation) to a given unity is also a unity.

Several elements (matrices) of a geometric algebra are said to be perpendicular if they anticommute. In this case, it is deduced that the norm of a linear combination of them fulfils the Pythagorean or pseudo-Pythagorean theorem. In a complete geometric algebra $M_{2^k\times2^k}(\mathbb{R})$ there are a maximum of $2k+1$ perpendicular unities, and $2k$ of them and their products induce the structure of Clifford algebra inside the matrix algebra (which we are calling geometric algebra) and form a basis of the algebra. Reflections are similarity transformations for all the even geometric algebras including the complete geometric algebras, while they are not in odd geometric algebras. The conditions a geometric algebra must fulfil for duality to be a similarity transformation are also given, showing that complete geometric algebras also have at least one Clifford structure where duality is a similarity transformation.

References

[1] J. W. Gibbs. On Multiple Algebra. *Proceedings of the American Association for the Advancement of Science*, **XXXV,** 37-66. 1886. Reproduced in *Scientific papers of J. Willard Gibbs* **II**: 91-117, Dover, New York, 1961.
[2] J. W. Gibbs. *Elements of Vector Analysis*. Privately printed, New Haven, 1881. *Ibidem* 17-90.
[3] J. W. Gibbs. Quaternions and Vector Analysis. *Nature*, **XLVIII**: 364-367, Aug. 17th 1893. *Ibidem* 173-181.
[4] J. W. Gibbs. On the Rôle of Quaternions in the Algebra of Vectors. *Nature*, **XLIII**: 511-513, April 2, 1891. *Ibidem* 155-160.
[5] D. Hestenes. *Space-Time Algebra*: 49, Gordon and Breach, New York, 1966.
[6] S. Gull, Ch. Doran, A. Lasenby: Dynamics. *Clifford (Geometric) Algebras*: 83, W. E. Baylis (ed.). Birkhäuser, Boston, 1996.
[7] R. González Calvet. *Treatise of Plane Geometry through Geometric Algebra*, Cerdanyola del Vallès, 2007.
[8] R. González Calvet. *El álgebra geométrica del espacio y tiempo*. http://www.xtec.cat/~rgonzal1/espacio.htm, 2011.
[9] P. A. M. Dirac. The Quantum Theory of the Electron. *Proc. Roy. Soc. Ser. A* **117**: 610, 1928.
[10] E. Majorana. Teoria simetrica dell'elettrone e del positrone. *Nuovo Cimento*, **5**: 171-184, 1937. Reproduced in *La Vita e l'opera di Ettore Majorana (1906-1938)*: 67-79, Academia Nazionale dei Lincei, Roma, 1966.
[11] P. Lounesto. *Clifford Algebras and Spinors*: 216, Cambridge University Press, Cambridge, 1997.
[12] P. Lounesto, *ibidem*: 215.
[13] C. B. Boyer. *A History of Mathematics*, 2nd edition: 570, Wiley, New York 1991.
[14] R. González Calvet. Applications of Geometric Algebra and the Geometric Product to Solve Geometric Problems. Talk held at the AGACSE 2010 conference (14th-16th July 2010), http://www.xtec.cat/~rgonzal1/proceedings.pdf.
[15] W. R. Hamilton: First Motive for naming the Quotient of two Vectors a Quaternion. *Elements of Quaternions*, 3rd ed., **I**: 110-113, Chelsea, New York, 1969.
[16] L. S. Pontriaguin. *Grupos continuos*: 170, Mir, Moscow, 1978.
[17] G. Peano. Gli elementi di calcolo geometrico, 1891. *Opere Scelte*, **III**: 41-69, ed. Cremonese, Roma, 1959.
[18] M. Riesz. *Clifford Numbers and Spinors*: 10, 1957-1958. Lecture Series **54**, Kluwer, Dordrecht, 1993.

[19] J. Snygg. *Clifford Algebra. A computational tool for physicists*: 6, Oxford Univ. Press, 1997.
[20] L. Dorst, D. Fontijne, S. Mann. *Geometric Algebra for Computer Science*: 168, Elsevier, Amsterdam, 2007.
[21] P. Renton. *Electroweak Interactions*: 35-36, Cambridge Univ. Press, Cambridge,1990.
[22] L. Landau, E. Lifchitz. *Physique théorique*, **4**, *Électrodynamique quantique*: 68, Mir, Moscow, 1989.
[23] S. Perlis. *Theory of Matrices*: 143, Dover, New York, 1991.
[24] G. Casanova, L'algèbre de Clifford et ses applications, *Advances in Applied Clifford Algebras* **12** : 18, 2002.
[25] J. M. Parra i Serra. Doctoral thesis, *Contribució a l'estudi de les àlgebres de Clifford*: 209, Universitat de Barcelona, 1984.