APPLICATIONS OF THE CLIFFORD-GRASSMANN ALGEBRA TO THE PLANE GEOMETRY

Ramon González Calvet, I.E.S. Pere Calders, campus U.A.B., E-08193 Cerdanyola del Vallès, Spain. rgonzal1@pie.xtec.es
http://ww.terra.es/personal/rgonzal1
Josep Manel Parra i Serra, Departament de Física Fonamental, U. B. , Avda. Diagonal 647, E-08028 Barcelona, Spain jmparra@ffn.ub.es
http://www.ffn.ub.es/~jmparra

The applications of Clifford algebra to plane geometry are shown in two different but complementary cases: the Euclidean and pseudo-Euclidean planes.

Euclidean plane

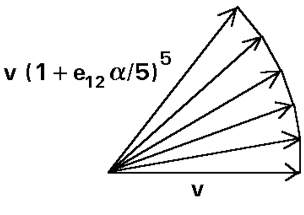
The algebra for the Euclidean plane is given by $e_1^2 = e_2^2 = 1$ and $e_{12} \equiv e_1 e_2 = -e_2 e_1$. Vectors are the first-degree elements $v = v_1\, e_1 + v_2\, e_2$ while the subalgebra of even degree elements is isomorphic to the complex number field: $z = a + b\, e_{12} \equiv a + bi$. Both types of quantities are integrated, without confusion, into a single algebraic structure of real dimension 4.

Geometric transformations of vectors are easily written with geometric algebra. In particular the rotation through an angle α can be expressed as:

$$v' = v\,(\cos\alpha + e_{12}\sin\alpha) = \left(\cos\frac{\alpha}{2} + e_{12}\sin\frac{\alpha}{2}\right)^{-1} v \left(\cos\frac{\alpha}{2} + e_{12}\sin\frac{\alpha}{2}\right)$$

While the first expression reproduces and explains the standard use of the complex numbers in plane geometry, only the second one generalises to every dimension and to geometric elements of any order. It introduces what has been called [1] the spinor 1/2 operators, first found by Rodrigues, Cayley and Hamilton for vectors in three dimensions. In general if z is a complex number with argument $\alpha / 2$, the rotation through an angle α is written as: $v' = z^{-1}\, v\, z$

Figure 1



Exactly the same formula applies to the reflection of a vector v with respect a direction along the vector u : $v' = u^{-1}\, v\, u$

The expression of rotations as a sequence of an even number of symmetry reflections follows naturally.

The exponential function of an imaginary argument (Euler and de Moivre's formulae) are both analytically obtained and geometrically interpreted as an instance of the exponentiation operator, that is, as the limit of the n-th power of the infinitesimal n-th root :

$$v \exp[\alpha\, e_{12}] = v \lim_{n \to \infty} \left(1 + \frac{\alpha\, e_{12}}{n}\right)^n$$

The figure 1 shows the resulting transformation for $n = 5$. At the limit $n \to \infty$, the resulting vector is obtained by a continuos rotation through an angle α, from where it follows the Euler's identity:

$$\exp\left[\alpha\, e_{12}\right] = \cos\alpha + e_{12}\sin\alpha$$

The usual inner and outer product of two vectors (their regressive and progressive Grassmann's products) may be written by means of their Clifford product as the symmetric and antisymmetric part respectively:

$$u \cdot v = \frac{u\,v + v\,u}{2} \qquad\qquad u \wedge v = \frac{u\,v - v\,u}{2}$$

The geometric (Clifford) product is exactly equivalent to the matrix product of the matrix representations of vectors and complex numbers (real 2×2 matrices). Then it is associative, distributive but not commutative, every element of the algebra having inverse except for special cases. Moreover, the product of three coplanar vectors has the *permutative* property: a vector in a product does not commute with neighbouring vectors but it can be permuted with a vector located two sites farther:

$$u\,v\,w = w\,v\,u$$

Due to its intrinsically geometric nature, the geometric algebra allows to solve geometric equations (which should be a main objective) without direct reference to coordinates, although the translation to these is always possible. As a sample, the formulas of the notable points of a triangle deduced in [2] are given. Denoting by P, Q and R the vertices of the triangle, the circumcentre O, the incentre I, and the orthocentre H are:

$$O = -\left(P^2\, QR + Q^2\, RP + R^2\, PQ \right)\left(2\, PQ \wedge QR \right)^{-1}$$

$$I = \frac{P\,\left|QR\right| + Q\,\left|RP\right| + R\,\left|PQ\right|}{\left|QR\right| + \left|RP\right| + \left|PQ\right|}$$

$$H = \left(P\;\; P \cdot QR + Q\;\; Q \cdot RP + R\;\; R \cdot PQ \right)\left(QR \wedge RP \right)^{-1}$$

where $\cdot$ symbolises the inner product, two letters together indicate a vector in the usual geometric notation, e. g. $PQ = Q - P$, and letters separated by a space indicate Clifford product (sometimes it is reduced to a product of vector by a scalar).

Pseudo-Euclidean plane

The algebra for the pseudo-Euclidean plane is usually given by $e_0^2 = -e_1^2 = 1$ and $e_{01} \equiv e_0 e_1 = -e_1 e_0$, which follow from imposing the hyperbolic norm $\left|v\right|^2 = v^2 = v_0^2 - v_1^2$ to the vector $v = v_0\, e_0 + v_1\, e_1$. Again the inner and outer product of two vectors are written in terms of the Clifford geometric product as:

$$u \cdot v = \frac{u\,v + v\,u}{2} \qquad\qquad u \wedge v = \frac{u\,v - v\,u}{2}$$

and like for Euclidean vectors, the product of hyperbolic vectors also fulfils the permutative property [2].

The so called hyperbolic rotation of a vector through an "angle" ψ (a Lorentz transformation of rapidity ψ or velocity $c \tanh \psi$ in physics)

$$\begin{cases} v_0' = v_0 \cosh\psi + v_1 \sinh \psi \\ v_1' = v_0 \sinh \psi + v_1 \cosh\psi \end{cases}$$

is again given by the action of the geometric algebra exponential of a bivector, being the hyperbolic functions consequence of the positive value of the square ${e_{01}}^2 = 1$. This Grassmann bivector is thus a geometric square root of 1 that cannot be reduced to the numbers 1 or –1. Thus we have for the finite rotation

$$\exp[\psi\, e_{01}] = \lim_{n \to \infty} \left(1 + \frac{\psi\, e_{01}}{n}\right)^n = \cosh\psi + e_{01} \sinh\psi$$

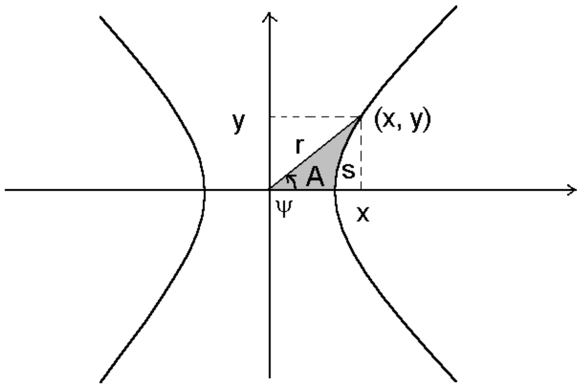
and for the vector transformation, written in the two-dimensional simplified way or in the general universal way $v' = z^{-1}\, v\, z$:

$$v' = v\,(\cosh\psi + e_{01} \sinh \psi) = \left(\cosh\frac{\psi}{2} + e_{01} \sinh\frac{\psi}{2}\right)^{-1} v \left(\cosh\frac{\psi}{2} + e_{01} \sinh\frac{\psi}{2}\right)$$

Following the limit $n \to \infty$ process applied to the unitary vector along the first axis, we see that the resulting vector is obtained by a continuos hyperbolic rotation through a hyperbolic angle ψ, drawing an arc of equilateral hyperbola as shown in figure 2. If r is the hyperbolic radius of the hyperbola $x^2 - y^2 = r^2$ the hyperbolic angle is proportional to the pseudo-Euclidean arc length s of the hyperbola:

$$\psi = \frac{s}{r}$$

Figure 2



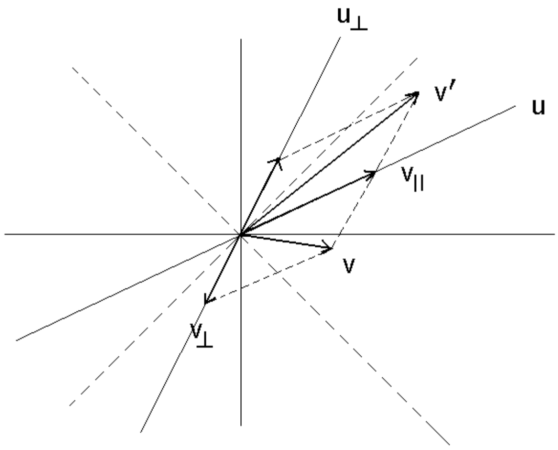
Seen as a Lorentz transformation this hyperbolic rotation preserves the modulus of momentum and position vectors. They are the mass of the particle and the space-time interval between two events. In the first case, all possible states of energy-momentum (*E, p*) for a particle of mass *m* correspond to all the points of the *m*-hyperbola:

$$E^2 - c^2 p^2 = (E')^2 - c^2 (p')^2 = m^2 c^4$$

Again the hyperbolic reflection of a vector *v* with respect a direction *u* is given by:

$$v' = u^{-1}\, v\, u = u^{-1}\,(\,v_{||} + v_{\perp})\, u = v_{||} - v_{\perp}$$

Figure 3



since it changes the sign of the component perpendicular to *u*. Two directions are perpendicular in the pseudo-Euclidean plane if the inner product of the vectors is zero. The geometric plot is shown in figure 3 where two perpendicular directions are seen as symmetrical with respect the bisector line of the first quadrant. It seems confusing, but the true trouble is not the plot but the paper, whose Euclidean proper geometry is subliminally captured by our eyes and assumed by our mind. We can get rid of these troubles using Minkowski's space-time, whose time-space points have a pseudo-Euclidean proper geometry, and looking at the paper figures as graphic representations of that essentially non-spatial reality: $c\,t\,e_0 + x\,e_1$. Also the energy-momentum components of a particle can be appropriately expressed by the hyperbolic vector $E\,e_0 + c\,p\,e_1$.

But from a mathematical point of view, to deal with these graphic representations as objects in itself is equally legitimate. Then we have got, combining the two plane geometric algebras, a unified account of both the circle and the hyperbola. An account that would surely be satisfactory to Leibniz's troubles about the still today dominant Cartesian geometry, expressed in a letter to Huygens in 1679 proposing a new geometric calculus: "Car premierement je puis exprimer parfaitement par ce calcul toute la nature ou définition de la figure (ce que l'algèbre ne fait jamais, car disant que $x^2 + y^2$ aeq. a^2 est l'équation du cercle, il faut expliquer par la figure ce que c'est *x* et *y*.)" [3],[4].

[1] David Hestenes, *New Foundations for Classical Mechanics,* Reidel, 1996.
[2] Ramon González, *Treatise of plane geometry through geometric algebra* (Cerdanyola del Vallès, 2000). http://campus.uab.es/~PC00018
[3] Josep M. Parra, "Geometric Algebra versus numerical cartesianism" in *Clifford Algebras and Their Applications in Mathematical Physics,* F. Brackx, R. Delanghe, H. Serras, eds. Kluwer, Dordrecht, 1993.
[4] Louis Couturat, *La logique de Leibniz,* PUF, Paris, 1901 (Georg Olms, 1961)