APPLICATIONS OF CLIFFORD-GRASSMANN ALGEBRA TO THE PLANE GEOMETRY

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The applications of Clifford algebra [**1**] (also called *geometric algebra*) to plane geometry are shown in two different but complementary cases: the Euclidean and pseudo-Euclidean planes.

The Euclidean plane

The geometric (or Clifford) product of two vectors is the addition of the scalar or inner product and the exterior or outer product [**2**], which results in a complex number:

$$\mathbf{v}\,\mathbf{w} = \mathbf{v}\cdot\mathbf{w} + \mathbf{v}\wedge\mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\left(\cos\alpha + \mathbf{e}_{12}\sin\alpha\right)$$

The square of a vector is identical to the square of its norm:

$$|\mathbf{v}|^2 = \mathbf{v}^2 = v_1^2 + v_2^2$$

and the geometric product of three vectors is associative. The basis vectors then fulfil:

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1$$

From the anticommutativity of perpendicular vectors, the area unity is identified with the imaginary unity of complex numbers:

$$\mathbf{e}_{12} = \mathbf{e}_1\,\mathbf{e}_2 = -\mathbf{e}_2\,\mathbf{e}_1 \qquad \mathbf{e}_{12}^2 = -1$$

The expressions of scalar and geometric products of two vectors written with geometric product:

$$\mathbf{v}\cdot\mathbf{w} = \frac{\mathbf{v}\,\mathbf{w} + \mathbf{w}\,\mathbf{v}}{2} \qquad \mathbf{v}\wedge\mathbf{w} = \frac{\mathbf{v}\,\mathbf{w} - \mathbf{w}\,\mathbf{v}}{2}$$

are ready to any algebraic manipulation with geometric algebra.

A rotation of a vector **v** through an angle α is described as a product of the vector **v** and the unitary complex number **z** [**3**, **4**]:

[**1**] Pertti LOUNESTO, *Clifford Algebras and Spinors*. London Mathematical Society Lecture Note Series **239**, Cambridge Univ. Press (Cambridge, 1997).
[**2**] David HESTENES, *New Foundations for Classical Mechanics*, ed. by Alwyn Van der Merwe, Reidel Publ. Company (Dordrecht, 1986) p. 30.
[**3**] Hermann GRASSMANN, *A new branch of mathematics: the "Ausdehnungslehre" and other works*, translation by Lloyd C. Kannenberg, Open Court (Chicago, 1995), p. 13.
[**4**] Giuseppe PEANO, *Gli elementi di calcolo geometrico* (1891), collected in *Opere Scelte* **III**, ed. Cremonese (Roma, 1959), p. 54.

$$\mathbf{v}' = \mathbf{v}\,\mathbf{z} \qquad \mathbf{z} = \cos\alpha + \mathbf{e}_{12}\sin\alpha$$

which has allowed us to prove in an algebraic way that the addition of distances from a point to the three vertices of a triangle is minimal for the Fermat point [**5**, p. 77].

The reflection of a vector **v** with respect to a line with direction vector **u** is written in geometric algebra as [**6**]:

$$\mathbf{v}' = \mathbf{u}^{-1}\,\mathbf{v}\,\mathbf{u}$$

The use of the algebraic properties of geometric algebra enables us to solve geometric equations and to obtain useful formulae as for the notable points of a triangle. For instance, the equations of the circumcentre O [**5**, p.71] and the orthocentre H [**5**, p.75] of a triangle ΔPQR are:

$$O = -\left(P^2\,\overrightarrow{QR} + Q^2\,\overrightarrow{RP} + R^2\,\overrightarrow{PQ}\right)\left(2\,\overrightarrow{PQ}\wedge\overrightarrow{QR}\right)^{-1}$$

$$H = \left(P\cdot\overrightarrow{QR}\,P + Q\cdot\overrightarrow{RP}\,Q + R\cdot\overrightarrow{PQ}\,R\right)\left(\overrightarrow{QR}\wedge\overrightarrow{RP}\right)^{-1}$$

Note that the product between parentheses is a geometric product in both cases. When calculating the direction of Euler's line we find a formula containing triple geometric products of the sides of the triangle ΔPQR:

$$\overrightarrow{OH} = -\left(\mathbf{a}\,\mathbf{b}\,\mathbf{c} + \mathbf{b}\,\mathbf{c}\,\mathbf{a} + \mathbf{c}\,\mathbf{a}\,\mathbf{b}\right)\left(2\,\mathbf{a}\wedge\mathbf{b}\right)^{-1} \qquad \mathbf{a} = \overrightarrow{PQ} \qquad \mathbf{b} = \overrightarrow{QR} \qquad \mathbf{c} = \overrightarrow{RP}$$

Pseudo-Euclidean plane

For the pseudo-Euclidean plane, the space-like unit vector has positive square while the time-like unit vector has negative square:

$$\mathbf{e}_0^2 = -1 \qquad \mathbf{e}_1^2 = 1 \qquad \mathbf{e}_{01} = \mathbf{e}_0\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_0$$

Then, the square of a vector is also equal to the square of its norm corresponding to a pseudo-Euclidean metric:

$$|\mathbf{v}|^2 = \mathbf{v}^2 = -v_0^2 + v_1^2$$

The product of two hyperbolic vectors yields a hyperbolic number:

$$\mathbf{v}\,\mathbf{w} = |\,\mathbf{v}\,|\,|\,\mathbf{w}\,|\left(\cosh\psi + \mathbf{e}_{01}\sinh\psi\right)$$

where ψ is the hyperbolic angle between **v** and **w**. The norm $|\,\mathbf{z}\,|$ of a hyperbolic number **z** is also pseudo-Euclidean:

[**5**] Ramon GONZÁLEZ CALVET, *Treatise of Plane Geometry through Geometric Algebra* (Cerdanyola del Vallès, 2007).
[**6**] William E. BAYLIS, ed., *Clifford (Geometric) Algebras*, Birkhäuser (Boston, 1996), p. 11.

$$\mathbf{z} = a + b\,\mathbf{e}_{01} \qquad \left|\mathbf{z}\right|^2 = \mathbf{z}\,\mathbf{z}^* = a^2 - b^2$$

A hyperbolic rotation (relativistic Lorentz transformation) is then written in the same way as Euclidean rotations by multiplying the hyperbolic vector on the left by a unitary hyperbolic number:

$$\mathbf{v}' = \mathbf{v}\,\mathbf{z} \qquad \mathbf{v} = v_0\mathbf{e}_0 + v_1\mathbf{e}_1 \qquad \mathbf{z} = \cosh\xi + \mathbf{e}_{01}\sinh\xi$$

where the hyperbolic argument ξ is related to the relative velocity V of both inertial frames through:

$$\xi = \arg\tanh\frac{V}{c}$$

where c is the light celerity. ξ is also proportional to the arc length of the equilateral hyperbola having radius $\left|\mathbf{v}\right|$ which touches the extremes of $\mathbf{v}$ and $\mathbf{v}'$:

$$\xi = \frac{\Delta s}{\left|\mathbf{v}\right|}$$

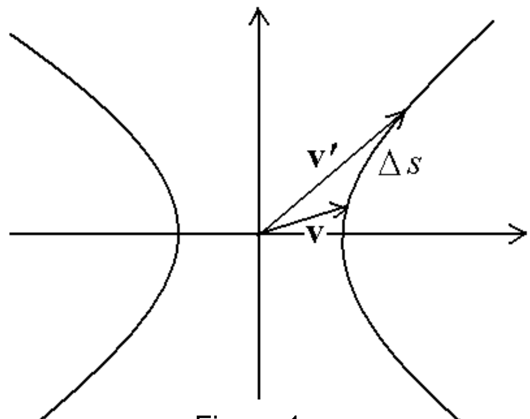


Figure 1

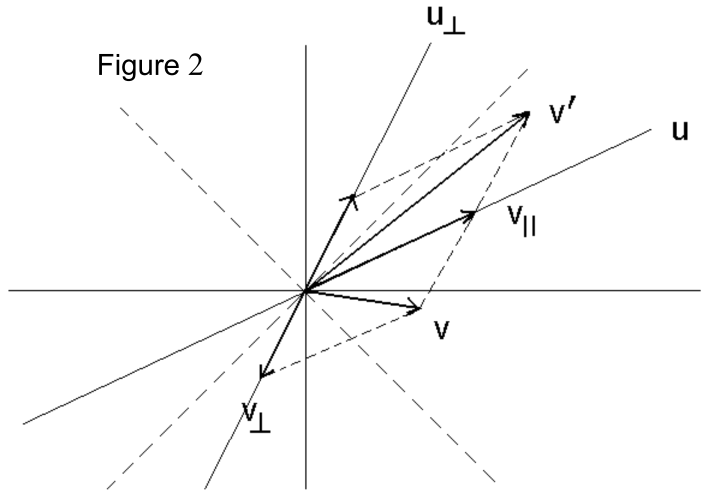
An axial symmetry of a hyperbolic vector **v** with respect to the direction **u** in the pseudo-Euclidean plane is written again as [**5**, p. 159]:

$$\mathbf{v}' = \mathbf{u}^{-1}\mathbf{v}\,\mathbf{u}$$

since it changes the sign of the component perpendicular to **u**:

$$\mathbf{v}' = \mathbf{u}^{-1}\left(\mathbf{v}_{||} + \mathbf{v}_{\perp}\right)\mathbf{u} = \mathbf{v}_{||} - \mathbf{v}_{\perp}$$

Figure 2



The geometric plot of an axial symmetry is shown in fig. 2, where two perpendicular directions such as **u** and $\mathbf{u}_{\perp}$ are seen by our eyes as symmetrical with respect to the bisector line in the first quadrant. From an early step of our life, our mind captures and processes the Euclidean properties of the room space, so that a plot on a flat paper seen by our eyes at a later age is subliminally interpreted by our mind as having Euclidean nature. We should properly interpret fig. 2 as a plane of Minkowski's space-time, although as Einstein showed our mind has trouble assuming relativistic concepts, owing to the very small velocities with which we move in our neighborhood.

The geometric algebra of the pseudo-Euclidean plane also gives the proof of some trigonometry theorems. Every triangle in the pseudo-Euclidean plane fulfils the law of hyperbolic sines:

$$\frac{|\mathbf{a}|}{\sinh\alpha} = \frac{|\mathbf{b}|}{\sinh\beta} = \frac{|\mathbf{c}|}{\sinh\gamma}$$

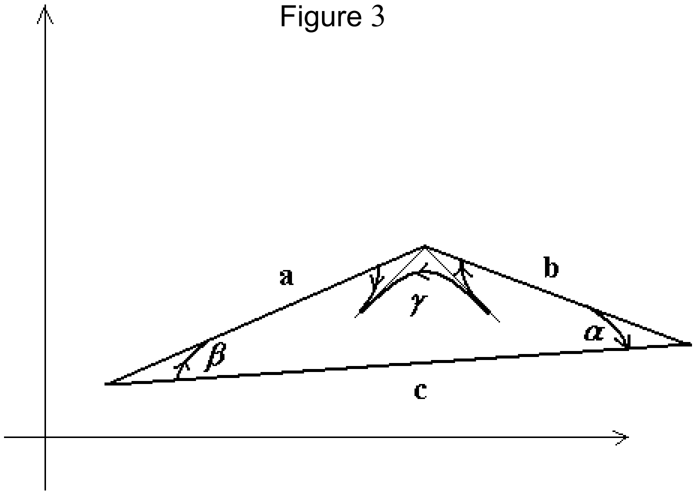
the law of hyperbolic cosines:

$$\mathbf{a}^2 = \mathbf{b}^2 + \mathbf{c}^2 - 2|\mathbf{b}||\mathbf{c}|\cosh\alpha$$

and the law of hyperbolic tangents:

$$\frac{|\mathbf{a}|+|\mathbf{b}|}{|\mathbf{a}|-|\mathbf{b}|} = \frac{\tanh\frac{\alpha+\beta}{2}}{\tanh\frac{\alpha-\beta}{2}}$$

Figure 3



Finally, two hyperbolic triangles are said to be directly similar [**7**] if their sides are geometrically proportional, that is, if the hyperbolic numbers being geometric quotients of the corresponding sides are equal [**5**, p.167]. In figure 4, the triangles $\Delta PRS'$ and $\Delta PSR'$ are similar so that we have:

$$PR'\,PS^{-1} = PR^{-1}PS'$$

By multiplying by *PR* on the left and *PS* on the right we obtain:

$$PR\,PR' = PS'\,PS$$

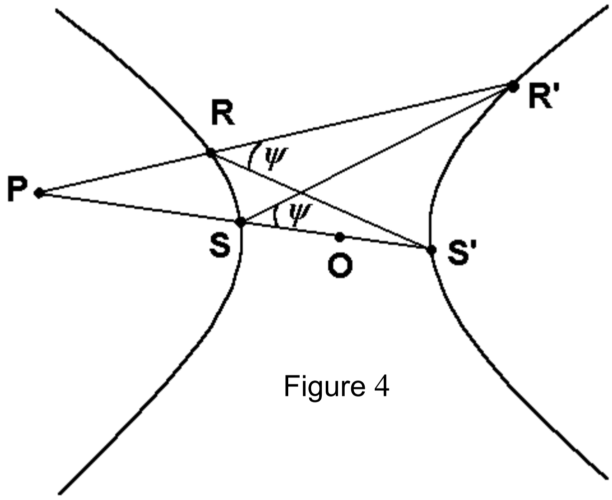


Figure 4

Since the products are of proportional vectors they are commutative: the product of distances from a point *P* to the points of intersection of a line passing through *P* and the equilateral hyperbola $x^2 - y^2 = r^2$ is constant independently of the chosen line. We call this product the *power of a point with respect to a hyperbola* of constant radius *r*. Finally, let us see that the power of a point is found by the substitution of its coordinates into the Cartesian equation of the hyperbola:

$$PS\,PS' = (PO + OS)(PO + OS') = PO^2 + OS\,OS' = x_P^2 - y_P^2 - r^2$$

[**7**] William Rowan HAMILTON (*Elements of Quaternions* [1869], ed. by Charles Jasper Joly, 3rd edition, Chelsea Publishing Company [N. Y., 1969], vol. I, p. 115.) stated that two similar triangles with sides *a*, *b*, *c* and *a*′, *b*′, *c*′ in the same plane of the Euclidean space are similar if and only if the quaternions obtained as quotients of two corresponding sides are equal ($a'\,a^{-1} = b'\,b^{-1}$).