APPLICATIONS OF GEOMETRIC ALGEBRA AND THE GEOMETRIC PRODUCT TO SOLVE GEOMETRIC PROBLEMS*

Ramon González Calvet

Institut Pere Calders, campus Universitat Autònoma de Barcelona s/n
08193 Cerdanyola del Vallès, Spain
rgonzalezcalvet@gmail.com

Introduction

In his *characteristica geometrica* [1, 2], Leibniz imagined and aimed to find an intrinsic geometric language to study geometry and to perform directly geometric operations in a way different from Cartesian coordinates. Afterwards, Möbius [3] introduced the barycentric calculus, from which Grassmann developed the extension theory. In 1846, Grassmann won the prize offered by the Fürstlich Jablonowski'schen Gesellschaft in Leipzig to whom was capable of developing Leibniz's idea with his memoir *Geometrische Analyse* [4, p.315]. While in his *Die Lineale Ausdehnungslehre* [4, p. 9] Grassmann introduced the inner and outer products of vectors, at the same time Hamilton discovered quaternions (1843), which he defined as geometric quotients of vectors [5]. Some years later, Clifford defined the geometric product [6] as the combination of Grassmann's inner and outer products. All these works conformed geometric algebra.

However, what is exactly geometric algebra and how is it understood? Peano, who had analyzed Grassmann's work deeply [7], already had a unified vision of geometric algebra [8, p.170]:

> "Indeed these various methods of geometric calculus do not at all contradict one another. They are various parts of the same science, or rather various ways of presenting the same subject by several authors, each studying it independently of the others."

As thought by Leibniz, geometric algebra has many evident advantages in comparison with Cartesian geometry. According to Peano [9, p. 169], who studied the foundations of geometry as much as those of arithmetic:

> "The geometric calculus differs from the Cartesian geometry in that whereas the latter operates analytically with coordinates, the former operates directly on the geometrical entities".

The main aim of geometric algebra should be to solve geometric problems. With this goal, Gibbs and Heaviside transformed quaternion calculus into vector analysis [9, 10], a non-associative and incomplete algebra of vectors. The difficulties of Cartesian geometry for solving many geometric problems were early pointed out by Gibbs [11]:

> "And the growth in this century of the so-called synthetic as opposed to analytical geometry seems due to the fact that by the ordinary analysis

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> geometers could not easily express, except in a cumbersome and unnatural manner, the sort of relations in which they were particularly interested".

In my opinion, however, geometric algebra is more than the Clifford algebra of a vector space [12, p.vi]:

> "The *geometric algebra* is the tool that allows us to study and solve geometric problems in a simpler and more direct way than purely geometric reasoning, that is, by means of the algebra of geometric quantities instead of *synthetic* geometry. In fact, the geometric algebra is the Clifford algebra generated by Grassman's outer product on a vector space, although for me, the geometric algebra is also the art of stating and solving geometric equations –which correspond to geometric problems- by isolating the unknown geometric quantity using the algebraic rules of vector operations (such as the associative, distributive and permutative properties)."

Therefore, geometric algebra is the new *Ars Magna*, since it is the most powerful and exclusive tool to solve geometric problems, as it will be shown below. Geometric algebra widens the field of application of the symbolic algebra of the Renaissance from real and complex quantities to vector entities. It is the algebra of the XXI century. The dream of Leibniz [13] has finally been accomplished.

Steps for solving geometric problems with geometric algebra

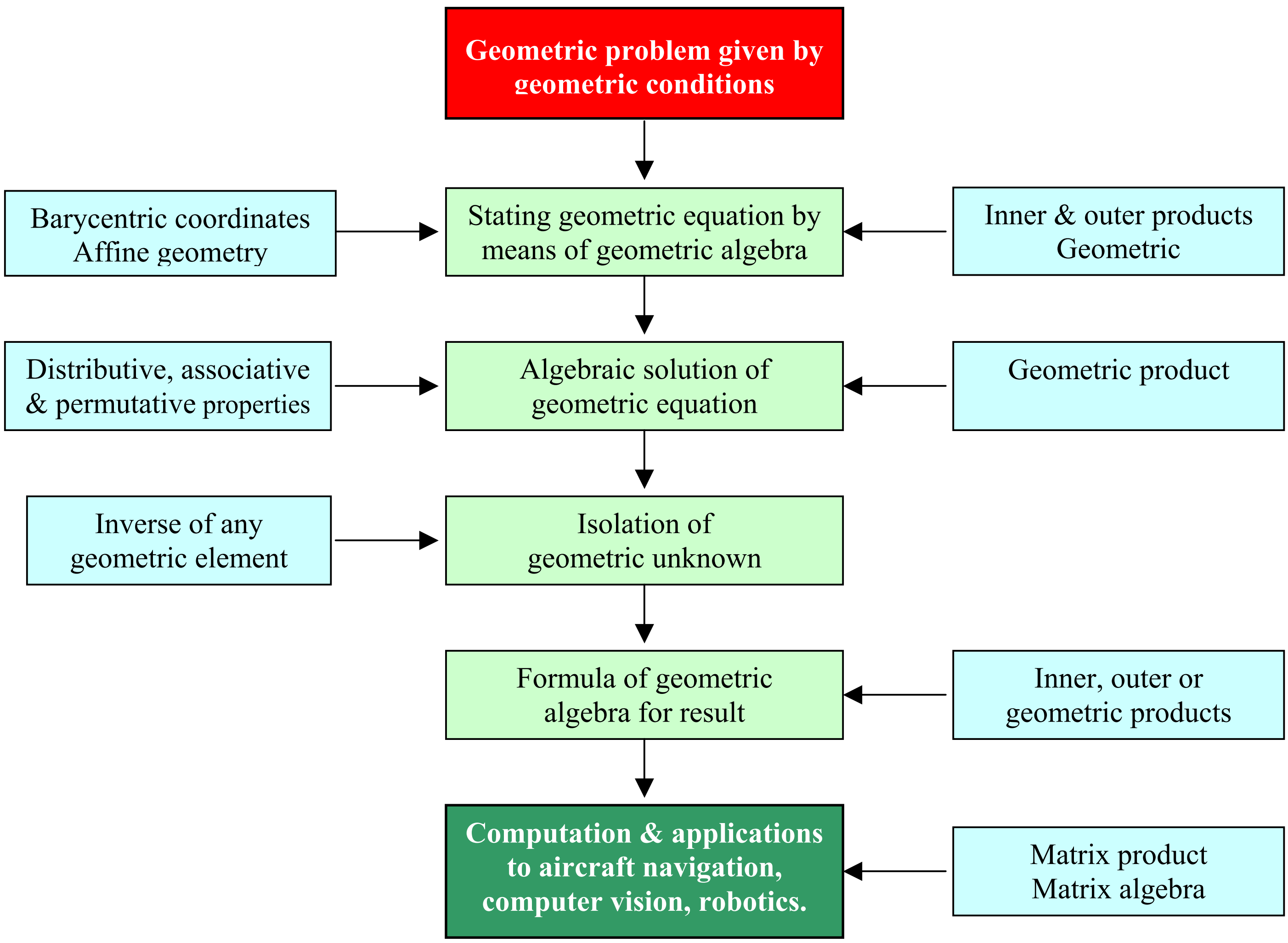
Let us see the steps that I follow to solve geometric problems with geometric algebra. First of all, we have a geometric problem given by geometric conditions (figure 1). We must state and write the geometric equation corresponding to this geometric problem. We make use of barycentric coordinates and affine geometry in order to write the relations among points and vectors. For instance, all the incidence problems (e. g. Desargues' theorem, Pappus' theorem ...) can be written in this way. In other types of problems the inner and outer products will be used, and, if some geometric transformation is involved, it can be written by means of the geometric product.

Once the equation is well stated, we proceed with its algebraic solution writing the inner and outer products by means of geometric product and applying the distributive, associative and permutative properties if needed. The geometric product allows to handle the algebraic equation without restrictions.

The goal of the process is to isolate the unknown geometric quantity (usually a vector) maybe with help of the inverse of some geometric element. Therefore, the importance of being a product with inverse.

Finally we arrive at a formula of geometric algebra for the unknown that may be written with inner, outer or geometric products depending on the symmetry of the problem. This formula can be applied to computation, aircraft navigation, computer vision or robotics, and it can be programmed by means of matrix algebra replacing the geometric elements by its matrix representation. In this case, we use the matrix product instead of the geometric product.

Figure 1. Steps for solving geometric problems with geometric algebra.



About the geometric product

In order to solve geometric equations we need a vector product with the following characteristics:

1) Associative property, which allows us the manipulation of geometric equations.
2) Distributive property for a suitable operation with vector addition.
3) The square of a vector equals the square of its norm.
4) The mixed associative property with scalars.

The geometric product deduced in this way fulfils the following additional properties:

1) There exists inverse of any vector, which allows us the isolation of a geometric unknown.
2) The product of two orthogonal vectors is anticommutative.
3) The product of proportional vectors is commutative.
4) $a\, b = a \cdot b + a \wedge b$.
5) Permutative property: $a\, b\, c = c\, b\, a$ if a, b, c are coplanar

Let us prove these properties.

1) Since the square of a vector equals the square of its norm, the inverse of a vector is proportional to this vector:

$$a^{-1} = \frac{a}{|a|^2} \qquad a^{-1} a = a\, a^{-1} = 1$$

2) If c is the vector addition of a and b ($c = a + b$) we have by the distributive property:

$$c^2 = (a + b)^2 = (a + b)(a + b) = a^2 + a\, b + b\, a + b^2$$

where we preserve the order of the factors because we do not know if the product has the commutative property. If a and b are orthogonal vectors, the Pythagorean theorem applies and then taking into account that the square of a vector is equal to the square of its modulus:

$$a \perp b \;\Rightarrow\; c^2 = a^2 + b^2 \;\Rightarrow\; a\, b + b\, a = 0 \;\Rightarrow\; a\, b = -\, b\, a$$

That is, the product of two orthogonal vectors is anticommutative.

3) On the other hand, if a and b are proportional vectors then their product is commutative:

$$a \,||\, b \;\Rightarrow\; b = k\, a,\; k \text{ real} \;\Rightarrow\; a\, b = a\, k\, a = k\, a\, a = b\, a$$

owing to the mixed associative property with scalars. If c is the addition of two vectors a, b with the same direction and sense, we have:

$$|\, c\,| = |\, a\,| + |\, b\,| \qquad c^2 = a^2 + b^2 + 2\, |\, a\,|\, |\, b\,|$$

4) The geometric product of two vectors is in the general case, by the distributive property, the addition of the scalar and exterior products:

$$a\, b = a\, (b_{||} + b_{\perp}) = a\, b_{||} + a\, b_{\perp} = |\, a\,|\, |\, b\,|\, (\cos\alpha + e_{12} \sin\alpha) = a \cdot b + a \wedge b$$

$a\, b_{\|} = |\, a\, |\, |\, b\, | \cos \alpha = a \cdot b$ inner or scalar product

$a\, b_{\perp} = |\, a\, |\, |\, b\, | (\cos \alpha + e_{12} \sin \alpha) = a \wedge b$ outer or exterior product

Writing the scalar and exterior products by means of the geometric product according to [14]:

$$a \cdot b = \frac{a\, b + b\, a}{2} \qquad\qquad a \wedge b = \frac{a\, b - b\, a}{2}$$

allows us the transposition and isolation of the unknown vector in geometric equations. The geometric product of two vectors is represented by the parallelogram they form.

5) Let us see the permutative property. In a geometric product of three or more vectors, one vector can be permuted with another vector two sites further without changing the product if all the vectors lie in the same plane. We may prove this by using the components that are parallel and perpendicular to the vector in the middle and by taking into account that proportional vectors commute while perpendicular vectors anticommute:

$$a\, b\, c = (a_{\|} + a_{\perp})\, b\, (c_{\|} + c_{\perp}) = a_{\|}\, b\, c_{\|} + a_{\|}\, b\, c_{\perp} + a_{\perp}\, b\, c_{\|} + a_{\perp}\, b\, c_{\perp}$$

$$= c_{\|}\, b\, a_{\|} + c_{\perp}\, a_{\|}\, b + c_{\|}\, b\, a_{\perp} + c_{\perp}\, b\, a_{\perp} = c\, b\, a$$

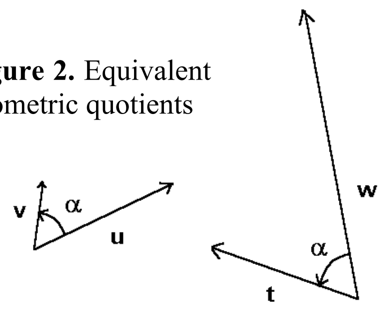
If the vectors are not coplanar then the difference of the product of three vectors minus the permutated product equals twice the volume of the parallelepiped formed by these vectors. We may prove this by using the component perpendicular to the plane containing the vectors on the extremes of the product, since the products by the coplanar component cancel each other by the permutative property:

$$a\, b\, c - c\, b\, a = a\, b_{\perp}\, c - c\, b_{\perp}\, a = b_{\perp} (-a\, c + c\, a) = -2\, b_{\perp}\, a \wedge c = 2\, a \wedge b \wedge c$$

About the geometric quotient

Hamilton explained what is the geometric quotient of two vectors in the section "First Motive for naming the Quotient of two Vectors a Quaternion" of the *Elements of Quaternions* (1866, [15]). If two pairs of vectors are the homologous sides of two similar triangles in the same plane (figure 2), then their geometric quotient is equal.

Figure 2. Equivalent geometric quotients



$$\left.\begin{array}{c} \alpha(u,v) = \alpha(w,t) \\ \dfrac{|u|}{|v|} = \dfrac{|w|}{|t|} \end{array}\right\} \Rightarrow \quad u\, v^{-1} = w\, t^{-1}$$

This geometric quotient needs four quantities to be given: the angle between vectors, the relative length, the inclination and the declination of the plane. Therefore, Hamilton defines the quotient of two vectors as a quaternion.

Geometric transformations

All the vector transformations can be written with geometric algebra by using the geometric elements that define them. For instance, a rotation (figure 3) is written as [16, 17]:

$$v' = q_{\alpha/2}^{-1} \; v \; q_{\alpha/2}$$

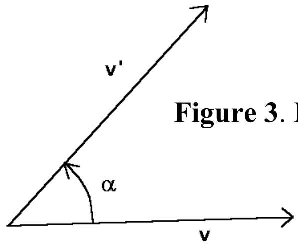
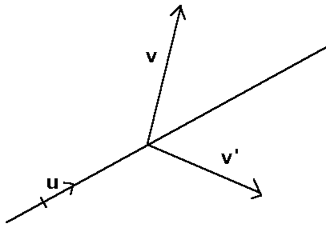


Figure 3. Rotation

where $q_{\alpha/2}$ is a quaternion of the three-dimensional space or a complex number in the plane whose argument is half the angle of rotation, although a briefer form exclusive for vectors *v* can be used in plane geometry [18, 19]:

$$v' = v \, 1_{\alpha} \, .$$

Figure 4. Axial symmetry



Axial symmetries (figure 4) with respect to a line having a direction vector *u* can also be written as:

$$v' = u^{-1} v \, u$$

The inversion with respect to a sphere (figure 5) with radius *r* is easily written by using the inverse of a vector:

$$v' = r^2 \; v^{-1}$$

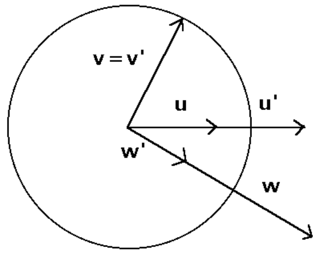


Figure 5. Inversion

And finally a reflection of a vector with respect to a plane with bivector *n* (gigure 6) is written as:

$$v' = -n^{-1} v \, n$$

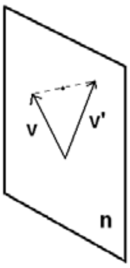


Figure 6. Reflection with respect to a plane

Examples of application of geometric algebra to solve geometric problems

Now, let us see several examples of application of geometric algebra and especially the geometric product to solve geometric problems.

Circumcentre of a triangle

The three bisectors of the sides of a triangle intersect in a unique point that is the centre of the circumscribed circle (figure 7), the circumcentre O, so that:

$$OP^2 = OQ^2 = OR^2 = d^2$$

where d is the radius of the circumscribed circle. The solution to these equations obtained by means of geometric algebra is [12, p.71]:

$$O = -\left(P^2\ QR + Q^2\ RP + R^2\ PQ\right)\left(2\ PQ \wedge QR\right)^{-1}$$

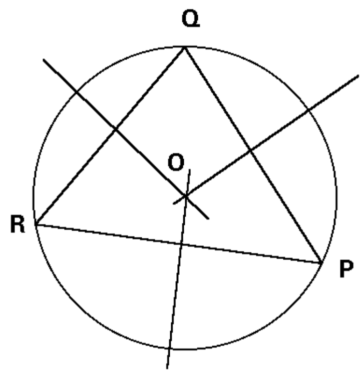


Figure 7. Bisectors of the sides and circumcentre

Note the geometric quotient of a vector by an imaginary number.

Orthocentre of a triangle

The three altitudes of a triangle ΔPQR intersect in a unique point called the orthocentre H of the triangle (figure 8). Since H belongs to the altitude perpendicular to QR that passes through the vertex P, H is obtained from P by the addition of a vector that is perpendicular to QR. It is obtained by multiplying QR by an imaginary unknown number:

$$H = P + z\, QR \qquad z \text{ imaginary}$$

Likewise, H also belongs to the altitude perpendicular to the base RP that passes through Q so that H is obtained from Q by an addition of a vector perpendicular to RP:

$$H = Q + t\, RP \qquad t \text{ imaginary}$$

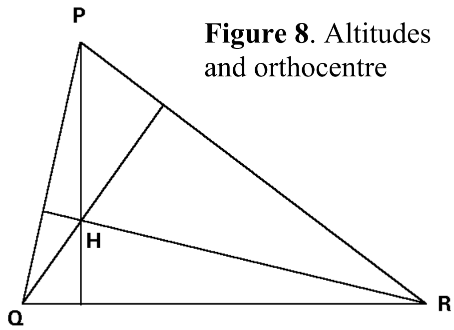


Figure 8. Altitudes and orthocentre

By solving this equation system, we arrive at a formula of the orthocentre that also contains a geometric quotient of vector by an imaginary number [12, p. 75].

$$H = \left(P\ P \cdot QR + Q\ Q \cdot RP + R\ R \cdot PQ\right)\left(QR \wedge RP\right)^{-1}$$

Vector of the Euler line of a triangle

By subtraction of the orthocentre from the circumcentre we arrived at the formula of the vector of the Euler line of a triangle [20]:

$$\begin{aligned} OH = H - O = \big(&P\ R\ P - P\ Q\ P + Q\ P\ Q - Q\ R\ Q + R\ Q\ R - R\ P\ R \\ &+ P^2\ R - P^2\ Q + Q^2\ P - Q^2\ R + R^2\ Q - R^2\ P\big)\left(2\ PQ \wedge QR\right)^{-1} \end{aligned}$$

$$OH = -[(Q-P)(R-Q)(P-R) + (R-Q)(P-R)(Q-P) + (P-R)(Q-P)(R-Q)\,](2\,PQ \wedge QR)^{-1}$$

By taking $a = PQ$, $b = QR$ and $c = RP$ the formula is written as:

$$OH = -(a\,b\,c + b\,c\,a + c\,a\,b)(2a \wedge b)^{-1}$$

where the triangle edges a, b and c are vectors and all the products are, of course, geometric products.

Centroid and circumcentre of a tetrahedron

The centroid G of a tetrahedron $ABCD$ is given by:

$$G = \frac{1}{4}(A + B + C + D)$$

The circumcentre O, the centre of the circumscribed sphere of the tetrahedron (figure 9), is equidistant from the four vertices so that:

$$OA^2 = OB^2 = OC^2 = OD^2 = d^2$$

where d is the radius of the circumscribed sphere. These equations lead to the equation system:

$$\left.\begin{aligned} O \cdot AB &= \frac{B^2 - A^2}{2} \\ O \cdot BC &= \frac{C^2 - B^2}{2} \\ O \cdot CD &= \frac{D^2 - C^2}{2} \end{aligned}\right\}$$

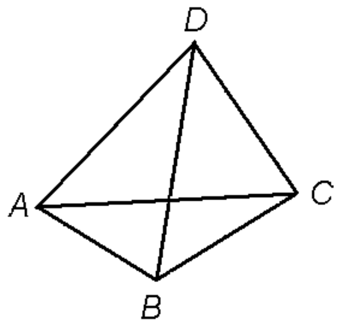


Figure 9. Tetrahedron

whose solution is:

$$O = \left[(B^2 - A^2)BC \wedge CD + (C^2 - B^2)CD \wedge AB + (D^2 - C^2)AB \wedge BC\right](2\,AB \wedge BC \wedge CD)^{-1}$$

or in a more symmetric form:

$$O = \left[-A^2\;BC \wedge CD + B^2\;CD \wedge DA - C^2\;DA \wedge AB + D^2\;AB \wedge BC\right](2\,AB \wedge BC \wedge CD)^{-1}$$

All the terms change the sign as well as the volume does under the cyclic permutation $A \to B$, $B \to C$, $C \to D$, $D \to A$, so that the formula remains unaltered.

Incentre of a tetrahedron

The incentre I lies at the same distance from the four faces of a tetrahedron. By analogy with the incentre of a triangle [12, p. 73]:

$$I = \frac{P\,|QR| + Q\,|RP| + R\,|PQ|}{|PQ| + |QR| + |RP|}$$

we infer that:

$$I = \frac{A\,|BC \wedge CD| + B\,|CD \wedge DA| + C\,|DA \wedge AB| + D\,|AB \wedge BC|}{|BC \wedge CD| + |CD \wedge DA| + |DA \wedge AB| + |AB \wedge BC|}$$

a result that is very easily checked by means of outer products:

$$r = \frac{|AB \wedge AC \wedge AI|}{|AB \wedge AC|} = \frac{|AC \wedge AD \wedge AI|}{|AC \wedge AD|} = \frac{|AD \wedge AB \wedge AI|}{|AD \wedge AB|} = \frac{|BC \wedge BD \wedge BI|}{|BC \wedge BD|}$$

where r is the radius of the inscribed sphere.

Monge point of a tetrahedron

The Monge point M is the intersection of the planes passing through the midpoint of an edge (e.g. CD) and perpendicular to the opposite edge (e.g. AB):

$$\begin{cases} \dfrac{MC + MD}{2} \cdot AB = 0 \\ \dfrac{MA + MD}{2} \cdot BC = 0 \\ \dfrac{MA + MB}{2} \cdot CD = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} M \cdot AB = \dfrac{C + D}{2} \cdot AB \\ M \cdot BC = \dfrac{A + D}{2} \cdot BC \\ M \cdot CD = \dfrac{A + B}{2} \cdot CD \end{cases}$$

The addition of any pair of equations leads to the equation for another plane so that the six planes intersect in a unique point M:

$$\begin{aligned} M = \big[\, (C + D) \cdot AB \;\; BC \wedge CD + (D + A) \cdot BC \;\; CD \wedge AB \\ + (A + B) \cdot CD \;\; AB \wedge BC \big] (2\; AB \wedge BC \wedge CD)^{-1} \end{aligned}$$

Euler line of a tetrahedron

By adding the equation systems for the circumcentre and the Monge point we observe the identity between the arithmetic mean of both points and the centroid:

$$\begin{cases} (O+M)\cdot AB = \dfrac{B^2-A^2}{2} + \dfrac{C+D}{2}\cdot AB = \dfrac{A+B+C+D}{2}\cdot AB \\ (O+M)\cdot BC = \dfrac{C^2-B^2}{2} + \dfrac{A+D}{2}\cdot BC = \dfrac{A+B+C+D}{2}\cdot BC \\ (O+M)\cdot CD = \dfrac{D^2-C^2}{2} + \dfrac{A+B}{2}\cdot CD = \dfrac{A+B+C+D}{2}\cdot CD \end{cases}$$

Therefore, the circumcentre, the Monge point and the centroid are collinear on the Euler line of a tetrahedron (figure 10):

$$\frac{O+M}{2} = \frac{A+B+C+D}{4} = G$$

Vector of the Euler line of a tetrahedron

The difference of both equation systems yields:

$$\begin{cases} OM\cdot AB = \dfrac{-DA+BC}{2}\cdot AB \\ OM\cdot BC = \dfrac{-AB+CD}{2}\cdot BC \\ OM\cdot CD = \dfrac{-BC+DA}{2}\cdot CD \end{cases}$$

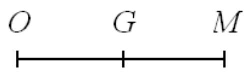


Figure 10. Euler line of a tetrahedron formed by the circumcentre, the centroid and the Monge point

whose solution is:

$$OM = \left(\frac{-DA+BC}{2}\cdot AB\ BC\wedge CD + \frac{-AB+CD}{2}\cdot BC\ CD\wedge AB + \frac{DA-BC}{2}\cdot CD\ AB\wedge BC\right)(AB\wedge BC\wedge CD)^{-1}$$

The geometric product allows us to write the direction vector of the Euler line of a tetrahedron in a more symmetric form:

$$OM = (AB\ BC\ CD\ DA - BC\ CD\ DA\ AB + CD\ DA\ AB\ BC - DA\ AB\ BC\ CD)(4AB\wedge BC\wedge CD)^{-1}$$

where all products are geometric products.

Fermat's theorem

An example of application of rotations is the Fermat theorem, which is easily proven by means of geometric algebra [12, p. 77]. Over the side of a triangle ΔABC we draw equilateral triangles. Introducing the complex number $t = \cos 2\pi/3 + e_{12} \sin 2\pi/3$, the rotation of AT through $2\pi/3$ is written as:

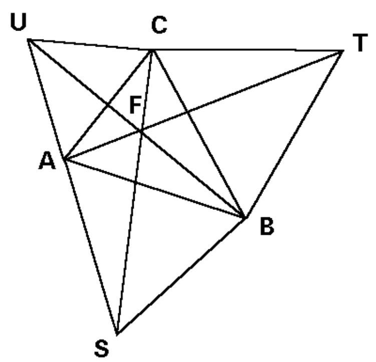


Figure 11. Fermat theorem

$AT\,t = (AC + CT\,)\,t = AC\,t + CT\,t = CU + BC = BU\,t$ = $\exp(2\pi e_{12}/3)$

Also: $CB = BU\,t$ and $AT = CS\,t$

so that CS, BU and AT have the same length, each of them is obtained from each other by successive rotations through $2\pi/3$ and they therefore intersect only in the Fermat point F. The addition of PA turned through $4\pi/3$, PB turned through $2\pi/3$ and PC is constant:

$$PA\,t^2 + PB\,t + PC = P'A\,t^2 + P'B\,t + P'C \quad \Leftrightarrow \quad PP'\,(t^2 + t + 1) = 0$$

because $t^2 + t + 1 = 0$. Hence, there is a unique point Q such that $PA\,t^2 + PB\,t + PC = QC$. For any point P, the three segments form a broken line that only becomes a straight line for the Fermat point F, whence it follows that the addition of the distances from F to the three vertices is minimal provided that no angle of the triangle is greater than $2\pi/3$.

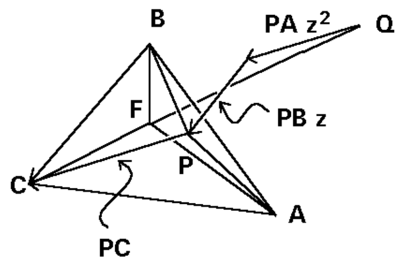


Figure 12. The addition of distances to the Fermat point is minimal.

Law of sines for spherical triangles

An angle of a spherical triangle is that formed by the central planes containing the sides of this angle, so that it is obtained from the bivectors of the sides:

$$\sin\alpha = \frac{|(A \wedge B) \times (A \wedge C)|}{|A \wedge B|\,|A \wedge C|}$$

where $\times$ is the antisymmetric product of two quaternions:

$$p \times q = -\frac{1}{2}(p\,q - q\,p)$$

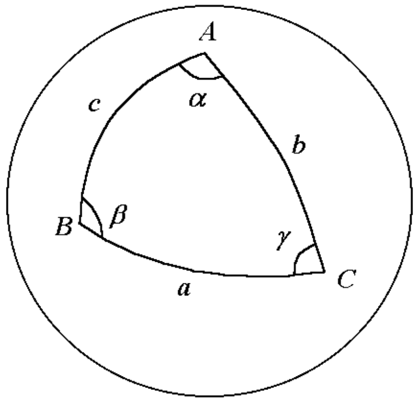


Figure 13. Spherical triangle

Now we write the products of the numerator using the geometric product [12, p. 173]:

$$\sin\alpha = \frac{|-(A\,B - B\,A)(A\,C - C\,A) + (A\,C - C\,A)(A\,B - B\,A)|}{8\,|A \wedge B|\,|A \wedge C|}$$

$$\sin\alpha = \frac{|-A\,B\,A\,C + A\,B\,C\,A + B\,A^2\,C - B\,A\,C\,A + A\,C\,A\,B - A\,C\,B\,A - C\,A^2\,B + C\,A\,B\,A|}{8\,|A \wedge B|\,|A \wedge C|}$$

Applying the permutative property to the suitable pairs of products, we have:

$$\sin\alpha = \frac{\left| 6\, A\wedge B\wedge C + 2\, A^{-1}\, A\wedge B\wedge C\, A \right|}{8\left|A\wedge B\right|\left|A\wedge C\right|} = \frac{\left|A\wedge B\wedge C\right|}{\left|A\wedge B\right|\left|A\wedge C\right|}$$

since the volume $A \wedge B \wedge C$ is a pseudoscalar, which commutes with all the elements of the algebra. Dividing by $\sin a = |\, B \wedge C\,|$, and so on we arrive at the law of sines [21]:

$$\frac{\sin\alpha}{\sin a} = \frac{\sin\beta}{\sin b} = \frac{\sin\gamma}{\sin c} = \frac{\left|A\wedge B\wedge C\right|}{\left|A\wedge B\right|\left|B\wedge C\right|\left|C\wedge A\right|}$$

Chasles' theorem

The projective cross ratio of a pencil of four lines can be written as a product and quotient of exterior products:

$$\left(X, ABCD\right) = \frac{XA\wedge XC\ XB\wedge XD}{XA\wedge XD\ XB\wedge XC}$$

The cross ratio is independent of the point X on the conic (Chasles' theorem, figure 14) and it is equal to the quotient of the half focal angles [12, p. 124]:

$$(X, ABCD) = \frac{\sin\dfrac{\angle AFC}{2}\sin\dfrac{\angle BFD}{2}}{\sin\dfrac{\angle AFD}{2}\sin\dfrac{\angle BFC}{2}}$$

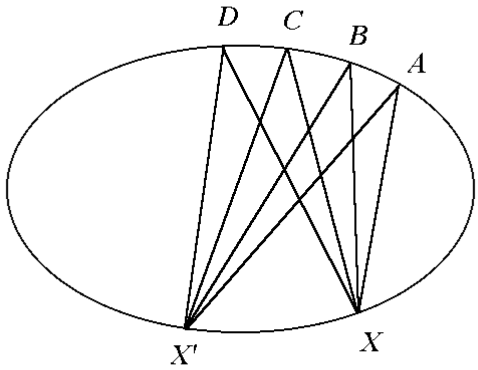


Figure 14. Chasles' theorem

where F is any focus of the conic. Let us see the application of Chasles' theorem to the problem of determining the point from where a photograph was taken. From five known references A, B, C, D and E in the photograph (figure 15) we calculate two cross ratios −($ABCD$) and ($ABCE$) in this example− and then we plot in the map (figure 16) the conics having these cross ratios. Both intersect in four points, three of which are A, B and C and the last intersection is the point X from where the photograph was taken [22].

Figure 15. Photograph of Barcelona harbour. A: Memorial to Columbus, B: End of the ceiling of the France station railroad terminal, C: West corner of the hotel Arts in the Vila Olímpica, D: The tower Jaume I of the aerial tramway over the harbour, E: The tower Sant Sebastià of the aerial tramway over the harbour.

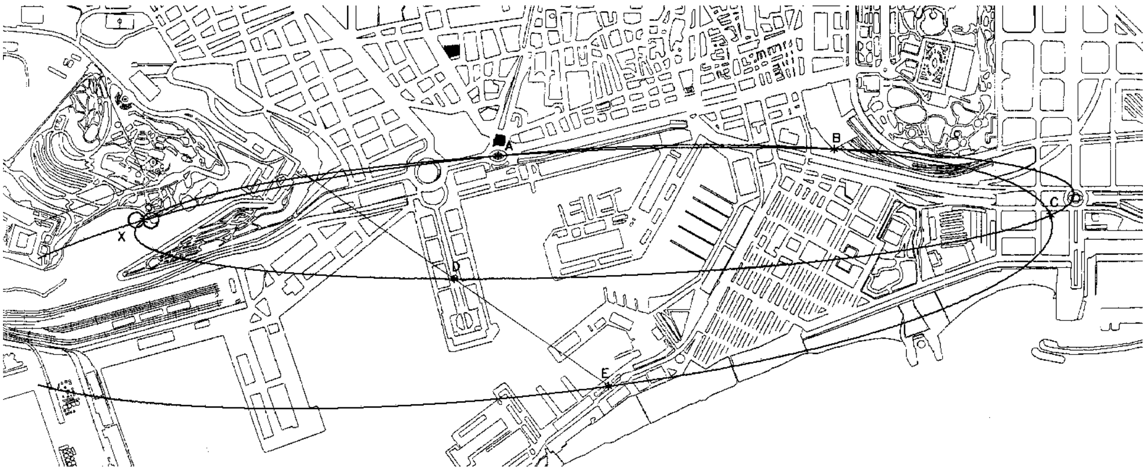


Figure 16. Map of Barcelona corresponding to the photograph in fig. 14 with the drawn conics passing through the points *ABCD* and *ABCE* that intersect in the point *X*, the Mirador de l'Alcalde in the Montjuic mountain.

We applied the formula of the projective cross ratio to the computation of the errors committed in the determination of the point from where a photograph was taken [23], and we found that the errors are independent of the pair of chosen conics.

Thank you very much for your attention. Have a nice stay in Amsterdam. We hope to see you in Barcelona soon!

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