TREATISE OF PLANE GEOMETRY
THROUGH GEOMETRIC ALGEBRA

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CONTENTS

First Part: The Euclidean vector plane and complex numbers

FIRST PART: THE EUCLIDEAN VECTOR PLANE AND COMPLEX NUMBERS

Points and vectors are the main elements of plane geometry. A *point* is conceived (but not defined) as a geometric element without extension, infinitely small, that has position and is located at a certain place in the plane. A *vector* is defined as an oriented segment, that is, a piece of a straight line having length and direction. A vector has no position and can be translated anywhere. It is usually called a *free* vector. If we place the end of a vector at a point, then its head determines another point so that a vector represents the translation from the first point to the second.

Taking into account the distinction between points and vectors, the portion of the book devoted to the plane Euclidean geometry has been divided into two parts. In the first one, vectors and their algebraic properties are studied, which is enough for many scientific and engineering branches. In the second part, points are introduced and the affine geometry is then studied.

All elements in geometric algebra (scalars, vectors, bivectors and complex numbers) are denoted by lowercase Latin characters and angles with Greek characters. Capital Latin characters will usually denote points in the plane. As you will see, the geometric product is not commutative so that fractions can only be written for real and complex numbers. Since the geometric product is associative, the inverse of a certain element on the left and on the right is the same, that is, there is a unique inverse for each element of the geometric algebra, which is indicated by the superscript $^{-1}$. Moreover, due to the associative property, all factors in a product are written without parentheses. In order to make the reading easy, neither theorems nor corollaries nor equations have been numerated. When a definition is introduced, the definite element is marked with italic characters that catch the reader's attention and help to find the definition once again.

1. EUCLIDEAN VECTORS AND THEIR OPERATIONS

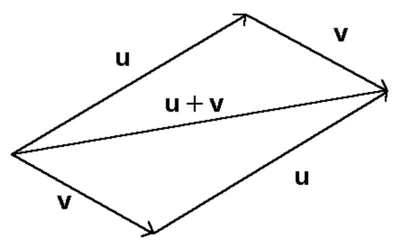
A vector is an oriented segment, having length and direction but no position, that is, it can be placed anywhere without changing its orientation. Vectors can represent many physical magnitudes such as force, velociity, and also geometric magnitudes such as translation.

We define two algebraic operations, the addition and the product of vectors, which generalise addition and multiplication of real numbers.

Vector addition

The *addition* of two vectors $u + v$ is defined as the vector going from the end of the vector u to the head of v when the head of u makes contact with the end of v (upper triangle in figure 1.1). Making the construction for $v + u$, that is, placing the end of u at the head of v (lower triangle in figure 1.1), we can see that the addition vector is the same.

Figure 1.1



Therefore, the vector addition has the commutative property:

$$u + v = v + u$$

and the parallelogram rule follows: the addition of two vectors is the diagonal of the parallelogram formed by both vectors. The associative property is the result of this definition because $(u+v)+w$ or $u+(v+w)$ is the vector closing the polygon formed by the three vectors, as shown in figure 1.2.

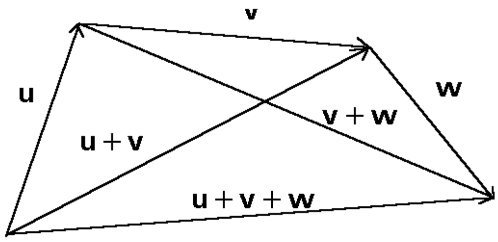


Figure 1.2

The neutral element of the vector addition is the *null vector*, which has no length. Hence the *opposite vector* of u is defined as the vector $-u$ with the same orientation but opposite direction, which added to the initial vector gives the null vector:

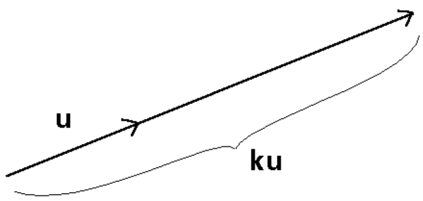
$$u + (-u) = 0$$

Product of a vector and a real number

The product of a vector and a real number (or *scalar*) k is defined as a vector with the same direction but whose length has been increased k times (figure 1.3). If the real number is negative, then the direction is opposite. The geometric definition implies the commutative property:

$$k\, u = u\, k$$

Figure 1.3



Two vectors u, v with the same direction are *proportional* because there is always a real number k such that $v = k\, u$, that is, k is the quotient of both vectors:

$$k = u^{-1}\, v = v\, u^{-1}$$

Two vectors with different directions are said to be *linearly independent*.

Product of two vectors

The product of two vectors will be called the *geometric product* in order to be distinguished from other vector products currently used. Nevertheless, I hope that these other products will play a secondary role when the geometric product becomes the most used, a near event this book will forward. The adjective "geometric" will then not be necessary.

We want the geometric product of two vectors to have the following properties:

1) To be distributive over vector addition:

$$u\,(\,v + w\,) = u\,v + u\,w$$

2) The square of a vector must be equal to the square of its length. By definition, the length (or *norm*) of a vector is a positive number and it is denoted by $|\,u\,|$:

$$u^2 = |\,u\,|^2$$

3) The mixed associative property must exist between the product of vectors and the product of a vector and a real number.

$$k\,(\,u\,v\,) = (\,k\,u\,)\,v = \;k\,u\,v$$

$$k\,(\,l\,u\,) = (\,k\,l\,)\,u = \;k\,l\,u$$

where *k, l* are real numbers and *u, v* vectors. Therefore, parentheses are not needed in this case.

These properties allow us to deduce the product. Let us suppose that c is the addition of two vectors a, b and let us calculate its square applying the distributive property:

$$c = a + b$$

$$c^2 = (\,a + b\,)^2 = (\,a + b\,)\,(\,a + b\,) = a^2 + a\,b + b\,a + b^2$$

We have to preserve the order of the factors because we do not know whether the product is commutative or not.

If a and b are orthogonal vectors, the Pythagorean theorem applies and then:

$$a \perp b \quad \Rightarrow \quad c^2 = a^2 + b^2 \quad \Rightarrow \quad a\,b + b\,a = 0 \quad \Rightarrow \quad a\,b = -\,b\,a$$

That is, <u>the product of two perpendicular vectors is anticommutative</u>.

If a and b are proportional vectors then:

$$a \parallel b \quad \Rightarrow \quad b = k\,a, \;\; k \text{ real} \quad \Rightarrow \quad a\,b = a\,k\,a = k\,a\,a = b\,a$$

because of the commutative and mixed associative properties of the product of a vector and a real number. Therefore <u>the product of two proportional vectors is commutative</u>. If c is the addition of two vectors *a, b* with the same direction and sense, we have:

$$|\,c\,| = |\,a\,| + |\,b\,|$$

$$c^2 = a^2 + b^2 + 2\,|\,a\,|\,|\,b\,|$$

$$a\,b = |\,a\,|\,|\,b\,| \qquad \angle(a, b) = 0$$

But if these vectors have opposite directions:

$$|\,c\,| = |\,a\,| - |\,b\,|$$

$$c^2 = a^2 + b^2 - 2\,|\,a\,|\,|\,b\,|$$

$$a\,b = -\,|\,a\,|\,|\,b\,| \qquad \angle(a, b) = \pi$$

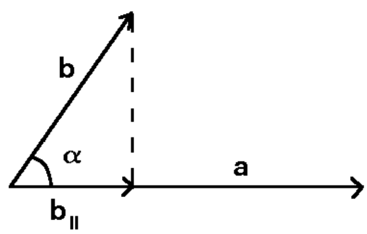
What is the product of two vectors with any directions? Due to the distributive property, the product is resolved into a product by the proportional component $b_{||}$ and another by the orthogonal component $b_{\perp}$:

$$a\,b = a\,(\,b_{||} + b_{\perp}\,) = a\,b_{||} + a\,b_{\perp}$$

The product of one vector by the proportional component of the other one is called the *inner product* (also *scalar product*) and denoted by a dot · (figure 1.4). Taking into account that the projection of b onto a is proportional to the cosine of the angle between both vectors, one finds:

$$a \cdot b = a\,b_{||} = |\,a\,|\,|\,b\,| \cos \alpha$$

Figure 1.4



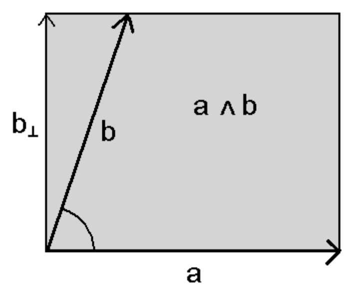
The inner product is always a real number. For example, the work made by some force acting on a body is the inner product of the force and the walked path. Since the commutative property has been deduced for the product of vectors with the same direction, the inner product is also commutative:

$$a \cdot b = b \cdot a$$

The product of one vector by the orthogonal component of the other vector is called the *outer product* (also *exterior product*) and it is denoted by the symbol $\wedge$:

$$a \wedge b = a\,b_{\perp}$$

Figure 1.5



The outer product represents the area of the parallelogram formed by both vectors (figure 1.5):

$$\left|\, a \wedge b \,\right| = \left|\, a\,b_{\perp} \,\right| = |a|\,|b|\,|\sin \alpha|$$

Since the outer product is a product of orthogonal vectors, it is anticommutative:

$$a \wedge b = -\,b \wedge a$$

Some examples of physical magnitudes that are outer products are angular momentum, torque and magnetic field.

When two vectors are permuted, the oriented angle is reversed. Then, its cosine remains unchanged while its sine changes the sign. Thus, the inner product is commutative while the outer product is anticommutative. Now we can rewrite the geometric product as the sum of both products:

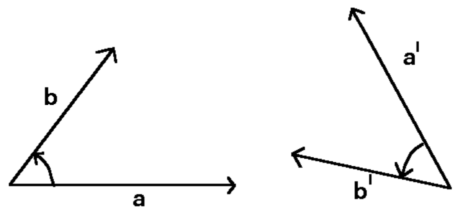
$$a\,b = a \cdot b + a \wedge b$$

From here, the inner and outer products can be written using the geometric product:

$$a \cdot b = \frac{a\,b + b\,a}{2}$$

$$a \wedge b = \frac{a\,b - b\,a}{2}$$

Figure 1.6



In conclusion, the geometric product of two proportional vectors is commutative whereas the product of two orthogonal vectors is anticommutative, just for the pure cases of outer and inner products. The outer, inner and geometric products of two vectors only depend upon the norms of both vectors and the angle between them. When both vectors are rotated preserving the angle they form, all three products are also preserved (figure 1.6).

What is the absolute value of the product of two vectors? Since the inner and outer products are linearly independent and orthogonal magnitudes, the norm of the geometric product must be calculated by means of a generalisation of the Pythagorean theorem:

$$a\,b = a \cdot b + a \wedge b \qquad \Rightarrow \qquad |\,a\,b\,|^2 = |\,a \cdot b\,|^2 + |\,a \wedge b\,|^2$$

$$|\,a\,b\,|^2 = |\,a\,|^2\,|\,b\,|^2\,(\cos^2\alpha + \sin^2\alpha) = |\,a\,|^2\,|\,b\,|^2$$

That is, the norm of the geometric product is the product of the norms of each vector:

$$|\,a\,b\,| = |\,a\,|\,|\,b\,|$$

Product of three vectors: associative property

It is demanded as the fourth property that the product of three vectors should be associative:

4) $\quad u\,(\,v\,w\,) = (\,u\,v\,)\,w = u\,v\,w$

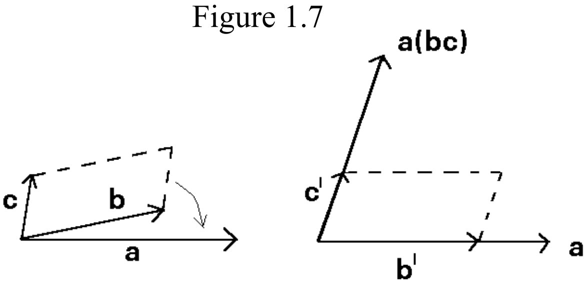
Hence we can remove parentheses in multiple products, and with the foregoing properties we can deduce how the product operates upon vectors.

We want to multiply a vector a by a product $b\,c$ of two vectors. We ignore the result of the product of three vectors with different orientations except when two adjacent factors are proportional. We have seen that the product of two vectors only depends on the angle between them. Therefore the parallelogram formed by b and c can be rotated until b has, in the new orientation, the same direction as a. If b' and c' are the vectors b and c with the new orientation (figure 1.7) then:

$$b\,c = b'\,c'$$

$$a\,(\,b\,c\,) = a\,(\,b'\,c'\,)$$

Figure 1.7



and by the associative property:

$$a\,(\,b\,c\,) = (\,a\,b'\,)\,c'$$

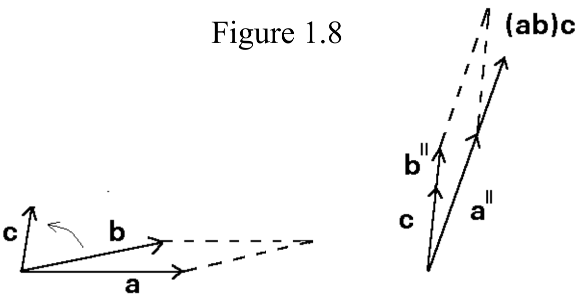
Since a and b' have the same direction, $a\,b' = |a|\,|b|$ is a real number and the triple product is a vector with the direction of c' whose length is increased by this amount:

$$a\,(\,b\,c\,) = |a|\,|b|\,c'$$

Consequently, the norm of the product of three vectors is the product of their norms:

$$|a\,b\,c| = |a|\,|b|\,|c|$$

Figure 1.8



On the other hand, we can first multiply a by b, and then we can rotate the parallelogram formed by both vectors until b has, in the new orientation, the same direction as c (figure 1.8). As a result:

$$(\,a\,b\,)\,c = a''(\,b''c\,) = a''\,|b|\,|c|$$

Although this geometric construction differs from the foregoing one, the figures clearly show that the triple product yields the same vector, as expected from the associative property. Moreover, we have:

$$(\,a\,b\,)\,c = a''\,|b|\,|c| = |c|\,|b|\,a'' = c\,b''\,a'' = c\,(\,b\,a\,)$$

That is, the triple product fulfils the *permutative* property:

$$a\,b\,c = c\,b\,a$$

Every vector can be permuted with a vector located two positions farther in a product, although it does not commute with the neighbouring vectors. The permutative property implies that any pair of vectors in a product separated by an odd number of vectors can be permuted. For example:

$$a\,b\,c\,d = a\,d\,c\,b = c\,d\,a\,b = c\,b\,a\,d$$

The permutative property is characteristic of the plane and it is also valid for the space whenever the three vectors are coplanar. This property is related to the fact that the product of complex numbers is commutative.

Product of four vectors

The product of four vectors can be deduced from the former reasoning. In order to multiply two pairs of vectors *a*, *b* and *c*, *d*, rotate the parallelogram formed by *a* and *b* until *b'* has the direction of *c*. The product is then the parallelogram formed by *a'* and *d*, but increased by the norm of *b* and *c*:

$$a\,b\,c\,d = a'\,b'\,c\,d = a'\,|\,b\,|\,|\,c\,|\,d = |\,b\,|\,|\,c\,|\,a'\,d$$

Now let us see the special case where $a = c$ and $b = d$. If both vectors *a, b* have the same direction, the square of their product is a positive real number:

$$a \parallel b \qquad (a\,b)^2 = a^2\,b^2 > 0$$

If both vectors are perpendicular, we must rotate the parallelogram through a right angle until *b'* has the same direction as *a* (figure 1.9). Then, *a'* and *b* are proportional but they have opposite directions. Therefore, the square of a product of two orthogonal vectors is always negative:

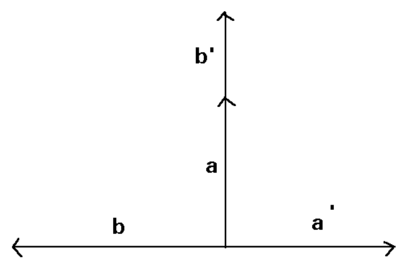


Figure 1.9

$$a \perp b \qquad (a\,b)^2 = a'\,b'\,a\,b = a'\,|\,b\,|\,|\,a\,|\,b = -a^2\,b^2 < 0$$

Likewise, the square of an outer product of any two vectors is also negative.

Inverse and quotient of two vectors

The *inverse* of a vector *a* is the vector whose multiplication by *a* gives the unity. Only vectors that are proportional have a real product. Hence the inverse vector has the same direction and inverse norm:

$$a^{-1} = \frac{a}{|\,a\,|^2} \qquad \Rightarrow \qquad a^{-1}\,a = a\,a^{-1} = 1$$

The quotient of two vectors is the product of one vector by the inverse of the other vector, which depends on the order of the factors because the product is not commutative:

$$a^{-1}\, b \neq b\, a^{-1}$$

Obviously, the quotient of proportional vectors with the same sense is equal to the quotient of their norms. When two vectors have different directions, their quotient can be represented by a parallelogram, which allows extending the concept of vector proportionality. We say that a is proportional to c as b is to d when their norms are proportional and the angle between a and c is equal to the angle between b and d[1]:

$$a\, c^{-1} = b\, d^{-1} \quad \Leftrightarrow \quad |a|\, |c|^{-1} = |b|\, |d|^{-1} \qquad \text{and} \qquad \angle(a, c) = \angle(b, d)$$

The parallelogram formed by a and b is then similar to that one formed by c and d. $\angle(a, c)$ is the angle of rotation from the first parallelogram to the second.

The inverse of a product of several vectors is the product of the inverses with an exchanged order, as you may easily deduce from the associative property:

$$(\, a\, b\, c\,)^{-1} = c^{-1}\, b^{-1}\, a^{-1}$$

Priority of algebraic operations

As in the algebra of real numbers, and in order to simplify the algebraic notation, we will apply the following priority to the vector operations explained above:

1) Parentheses, whose contents will be first operated.
2) Powers with any exponent (square, inverse, etc.).
3) Outer and inner products, which have the same priority but must be operated before geometric products.
4) Geometric products.
5) Additions.

As an example, some algebraic expressions are given with the simplified expression on the left side and its meaning using parentheses on the right side:

$$a \wedge b\; c \wedge d = (\, a \wedge b\,)\,(\, c \wedge d\,)$$

$$a^2\, b \wedge c + 3 = (\,(\, a^2\,)\,(\, b \wedge c\,)\,) + 3$$

$$a + b \cdot c\, d\, e = a + (\,(\, b \cdot c\,)\, d\, e\,)$$

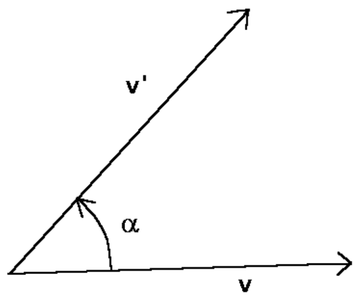
[1] Sir William Rowan Hamilton defined quaternions as quotients of two vectors in such a way that similar parallelograms located in the same plane in the three-dimensional space represent the same quaternion (*Elements of Quaternions*, posthumously edited in 1866, Chelsea Publishers 1969, vol. I, see p. 113 and fig. 34). In the vector plane, quaternions are reduced to a complex numbers. Quaternions were discovered by Hamilton on October 16th, 1843 before Clifford's geometric product (1878).

4. TRANSFORMATIONS OF VECTORS

Transformations of vectors are mappings of the vector plane onto itself. Those transformations preserving the norm of vectors, such as rotations and axial symmetries, are called *isometries* and those preserving angles between vectors are said to be *conformal*. Besides rotations and axial symmetries, inversions and dilations are also conformal transformations.

Rotations

Figure 4.1



A *rotation* through an angle α is the geometric operation consisting of turning a vector until it forms an angle α with the previous orientation. The positive direction of angles is counterclockwise (figure 4.1). Under rotations the norm of any vector is preserved. According to the definition of the geometric product, the multiplication of a vector v by a unit complex number with argument α produces a vector v' rotated through an angle α with respect to v.

$$v' = v\ 1_\alpha = v\ (\cos\alpha + e_{12}\sin\alpha)$$

This algebraic expression for rotations, when applied to a real or complex number instead of a vector, modifies its value. However, real numbers are invariant under rotations and the parallelograms can be turned without changing the complex number they represent. Therefore this expression for rotations, although it is useful for vectors, is not valid for complex numbers. In order to remodel it, we factorise the unit complex number into a product of two complex numbers with half the argument. According to the permutative property, we can permute the vector and the first complex number whenever writing the conjugate:

$$v' = v\ 1_\alpha = v\ 1_{\alpha/2}\ 1_{\alpha/2} = 1_{-\alpha/2}\ v\ 1_{\alpha/2} = (\cos\frac{\alpha}{2} - e_{12}\sin\frac{\alpha}{2})\ v\ (\cos\frac{\alpha}{2} + e_{12}\sin\frac{\alpha}{2})$$

The algebraic expression for rotations now found preserves complex numbers:

$$z' = 1_{-\alpha/2}\ z\ 1_{\alpha/2} = z$$

Let us calculate the rotation of the vector $4e_1$ through $2\pi/3$ by multiplying it by the unit complex number with this argument:

$$v' = 4\,e_1\left(\cos\frac{2\pi}{3} + e_{12}\sin\frac{2\pi}{3}\right) = 4\,e_1\left(-\frac{1}{2} + e_{12}\frac{\sqrt{3}}{2}\right) = -2\,e_1 + 2\sqrt{3}\,e_2$$

On the other hand, using the half angle $\pi/3$ we have:

$$v' = \left(\cos\frac{\pi}{3} - e_{12} \sin\frac{\pi}{3} \right) 4\, e_1 \left(\cos\frac{\pi}{3} + e_{12} \sin\frac{\pi}{3} \right)$$

$$= \left(\frac{1}{2} - e_{12} \frac{\sqrt{3}}{2} \right) 4\, e_1 \left(\frac{1}{2} + e_{12} \frac{\sqrt{3}}{2} \right) = -2\, e_1 + 2\sqrt{3}\, e_2$$

Using the expression of half angle, it is not necessary that the complex number have unit norm because:

$$z = |z|\, 1_{\alpha/2} \qquad\qquad z^{-1} = 1_{-\alpha/2}\, |z|^{-1}$$

Then, the rotation through an angle α can be written as:

$$v' = z^{-1}\, v\, z$$

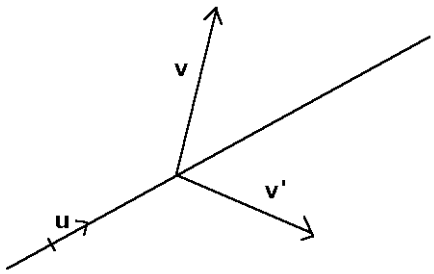
The composition of two successive rotations implies the product of both complex numbers, whose argument is the addition of the angles of both rotations.

$$v'' = 1_{-\beta/2}\, v'\, 1_{\beta/2} = 1_{-\beta/2}\, 1_{-\alpha/2}\, v\, 1_{\alpha/2}\, 1_{\beta/2} = 1_{-(\alpha+\beta)/2}\, v\, 1_{(\alpha+\beta)/2}$$

Axial symmetries

An *axial symmetry* (also called *reflection*) of a vector with respect to a direction is the geometric transformation that keeps constant the component with this direction and reverses the perpendicular component (figure 4.2). The product of proportional vectors is commutative and that of orthogonal vectors is anti-commutative. It is for this reason that the symmetric vector v' may be obtained from the multiplication of the vector v by the unit vector u of the symmetry axis on the left and right hand sides:

Figure 4.2



$$v' = u\, v\, u = u\, (\, v_{\|} + v_{\perp}\,)\, u = u\, v_{\|}\, u + u\, v_{\perp}\, u = v_{\|} - v_{\perp} \qquad \text{with} \quad u^2 = 1$$

where $v_{\|}$ and $v_{\perp}$ are the components of v respectively proportional and perpendicular to u.

Instead of the unit vector u, any vector d having the axis direction can be introduced into the expression for axial symmetries, whenever we write its inverse on the left side of the vector:

$$v' = \frac{d\, v\, d}{|\, d\,|^2} = d^{-1} v\, d$$

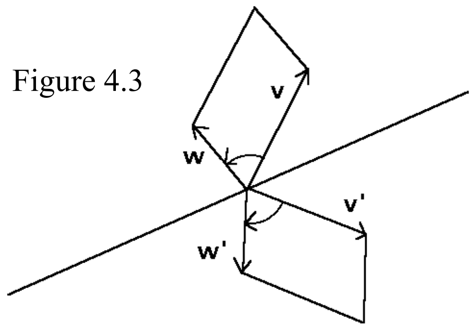
Although axial symmetries do not change the absolute value of the angle between two vectors, they change its sign. Under axial symmetries, real numbers remain invariant but complex numbers become conjugate because an axial symmetry generates a symmetric parallelogram (figure 4.3) and changes the sign of the imaginary part:

$z = a + b\, e_{12}$ $\qquad$ *a, b* real

$$z' = \frac{d\ (a + b\, e_{12})\, d}{|\, d\,|^2}$$

$$= \frac{d^2 a - d^2 b\, e_{12}}{|\, d\,|^2} = a - b\, e_{12} = z^*$$

Figure 4.3



For example, let us calculate the axial symmetry of the vector $3\ e_1 + 2\ e_2$ with respect to the direction $e_1 - e_2$. The resulting vector will be:

$$v' = \frac{1}{2}(\, e_1 - e_2\,)\,(\, 3\, e_1 + 2\, e_2\,)\,(\, e_1 - e_2\,) = \frac{1}{2}\ (\, e_1 - e_2\,)\,(\, 1 - 5\, e_{12}\,) = -\,2\, e_1 - 3\, e_2$$

Inversions

An *inversion* of radius *r* is the geometric transformation that maps every vector *v* onto $r^2\, v^{-1}$, that is, onto a vector with the same direction but with a norm equal to $r^2 /\ |\, v\,|$:

$v' = r^2\, v^{-1}$ $\qquad$ *r* real

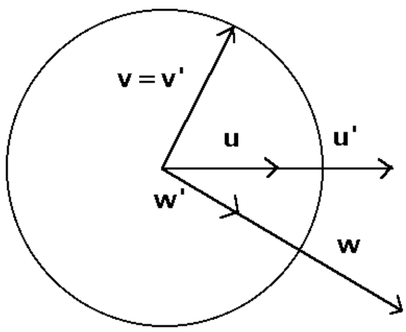


Figure 4.4

This operation is a generalisation of the inverse of a vector in geometric algebra (radius $r = 1$). It is called inversion of radius *r*, because all the vectors with norm *r*, whose heads lie on a circle with this radius, remain unchanged (figure 4.4). The vectors whose heads are placed inside the circle of radius *r* are transformed into vectors having the head outside and reciprocally.

Inversion transforms complex numbers into proportional complex numbers with the same argument (figure 4.5):

$v' = r^2\, v^{-1} \qquad w' = r^2\, w^{-1} \qquad z = v\, w$

$$z' = v'\, w' = r^4\, v^{-1}\, w^{-1} = r^4\, v\, w\, v^{-2}\, w^{-2} = \frac{r^4\, z}{|\, z\,|^2}$$

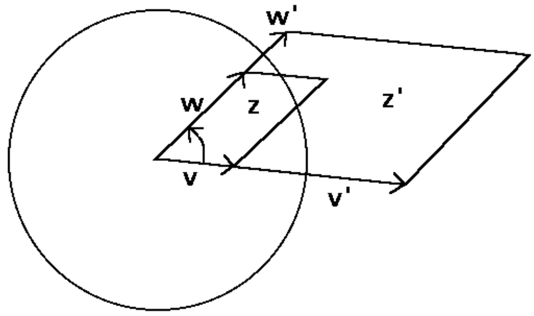
Dilations

A *dilation* is the geometric transformation that enlarges or shortens a vector, that is, it increases (or reduces) k times the norm of any vector while preserving its orientation. Dilation is simply the product by a real number k:

$$v' = k\, v \qquad k \text{ real}$$

If k is negative, the vector direction is reversed.

Figure 4.5



Most of the transformations of vectors that will be used in this book are combinations of these four elementary transformations. Many physical laws are invariant under some of these transformations. In geometry, from vector transformations we define transformations of points in the plane, indispensable for solving geometric problems.

Exercises

4.1 Calculate with geometric algebra what is the composition of an axial symmetry with a rotation.

4.2 Prove that the composition of two axial symmetries with respect to different directions is a rotation.

4.3 Consider the transformation under which every vector v multiplied by its transformed v' is equal to a constant complex z^2. Resolve it into elementary transformations.

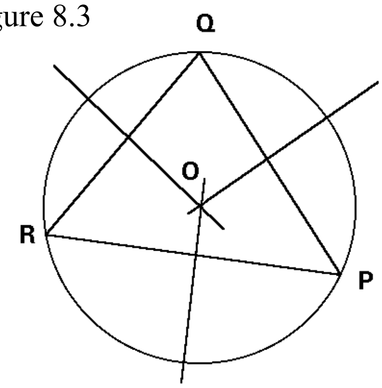
4.4 Apply a rotation through $2\pi/3$ to the vector $-3\, e_1 + 2\, e_2$ and find the resulting vector.

4.5 Find the axial symmetry of the former vector in the direction $e_1 + e_2$.

Perpendicular bisectors and circumcentre

The three perpendicular bisectors of the sides of a triangle meet at a unique point called *circumcentre,* the centre of the circumscribed circle. Every point on the perpendicular bisector of PQ is equidistant from P and Q. Analogously every point on the perpendicular bisector of PR is equidistant from P and R. The intersection O of both perpendicular bisectors is simultaneously equidistant from P, Q and R. Therefore O also belongs to the perpendicular bisector of QR and the three bisectors meet at a unique point. Since O is equally distant from the three vertices, it is the centre of the circumscribed circle. Let us use this condition in order to calculate the equation of the circumcentre:

Figure 8.3



$$OP^2 = OQ^2 = OR^2 = d^2$$

where d is the radius of the circumscribed circle. Using position vectors of each point we have:

$$(P - O)^2 = (Q - O)^2 = (R - O)^2$$

The first equality yields:

$$P^2 - 2\, P \cdot O + O^2 = Q^2 - 2\, Q \cdot O + O^2$$

By simplifying and arranging the terms containing O on the left hand side, we have:

$$2\,(\, Q - P\,) \cdot O = Q^2 - P^2$$

$$2\, PQ \cdot O = Q^2 - P^2$$

From the second equality we find an analogous result:

$$2\, QR \cdot O = R^2 - Q^2$$

Now we introduce geometric product instead of inner product into these equations:

$$PQ\;\; O + O\, PQ = Q^2 - P^2$$

$$QR\; O + O\; QR = R^2 - Q^2$$

By subtraction of the second equation multiplied on the right by PQ minus the first equation multiplied on the left by QR, we obtain:

$$PQ\;\; QR\;\; O - O\; PQ\;\; QR = PQ\;\; R^2 - PQ\;\; Q^2 - Q^2\;\; QR + P^2\;\; QR$$

By using the permutative property on the left hand side and simplifying the right hand side, we have:

$$PQ\; QR\; O - QR\; PQ\; O = P^2\; QR + Q^2\; RP + R^2\; PQ$$

$$2\,(PQ \wedge QR)\,O = P^2\,QR + Q^2\,RP + R^2\,PQ$$

Finally, multiplication by the inverse of the outer product on the left gives:

$$O = (2\,PQ \wedge QR)^{-1}\,(P^2\,QR + Q^2\,RP + R^2\,PQ)$$

$$= -\,(P^2\,QR + Q^2\,RP + R^2\,PQ)\,(2\,PQ \wedge QR)^{-1}$$

a formula suitable to calculate the coordinates of the circumcentre. For example, let us calculate the centre of the circle passing through the points $P(2, 2)$, $Q(3, 1)$ and $R(4, -2)$:

$$P^2 = 8 \qquad Q^2 = 10 \qquad R^2 = 20$$

$$QR = R - Q = e_1 - 3\,e_2 \qquad RP = P - R = -\,2\,e_1 + 4\,e_2 \qquad PQ = Q - P = e_1 - e_2$$

$$2\,PQ \wedge QR = -\,4\,e_{12}$$

$$O = -\,(8\,(e_1 - 3\,e_2) + 10\,(-2\,e_1 + 4\,e_2) + 20\,(e_1 - e_2))\frac{e_{12}}{4}$$

$$= -\,e_1 - 2\,e_2 = (-1, -2)$$

In order to deduce the circle radius, we take vector OP:

$$OP = P - O = P + (P^2\,QR + Q^2\,RP + R^2\,PQ)\,(2\,PQ \wedge QR)^{-1}$$

and extract the inverse of the area as common factor:

$$OP = (2\,P\;\;PQ \wedge QR + P^2\,QR + Q^2\,RP + R^2\,PQ)\,(2\,PQ \wedge QR)^{-1} =$$

$$= [\,2\,P\,(P \wedge Q + Q \wedge R + R \wedge P) + P^2\,QR + Q^2\,RP + R^2\,PQ\,]\,(2\,PQ \wedge QR)^{-1} =$$

$$= [\,P\,(P\,Q - Q\,P + Q\,R - R\,Q + R\,P - P\,R) + P^2\,(R - Q) + Q^2\,(P - R) +$$

$$+ R^2\,(Q - P)\,]\;\;(2\,PQ \wedge QR)^{-1}$$

Simplification gives:

$$OP = (P\,Q\,R - P\,R\,Q + P\,R\,P - P\,Q\,P + Q^2\,P - Q^2\,R + R^2\,Q - R^2\,P)\,(2\,PQ \wedge QR)^{-1}$$

$$= -\,(Q - P)\,(R - Q)\,(P - R)\,(2\,PQ \wedge QR)^{-1} = -\,PQ\;\;QR\;\;RP\,(2\,PQ \wedge QR)^{-1}$$

Analogously:

$$OQ = -\,QR\;\;RP\;\;PQ\,(2\,PQ \wedge QR)^{-1} \qquad\qquad OR = -\,RP\;\;PQ\;\;QR\,(2\,PQ \wedge QR)^{-1}$$

The radius of the circumscribed circle is the length of any of these vectors:

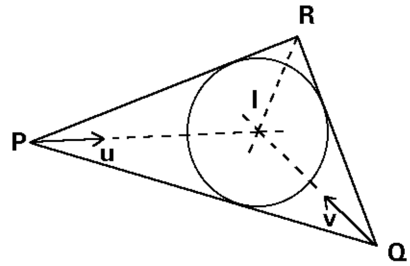
$$|OP| = \frac{|PQ|\,|QR|\,|RP|}{2\,|PQ \wedge QR|} = \frac{|PQ|}{2\sin QRP} = \frac{|QR|}{2\sin RPQ} = \frac{|RP|}{2\sin PQR}$$

where we find the law of sines.

Angle bisectors and incentre

The three bisectors of the angles of a triangle meet at a unique point called *incentre*. Any point on the bisector of the angle with vertex P is equidistant from sides PQ and PR (figure 8.4). Any point on the angle bisector passing through Q is also equidistant from sides QR and QP. Hence their intersection I is simultaneously equidistant from the three sides, that is, I is unique, and it is the centre of the circle inscribed in the triangle.

Figure 8.4



In order to calculate the equation of the angle bisector passing through P, we take the sum of the unit vectors of both adjacent sides:

$$u = \frac{PQ}{|PQ|} + \frac{PR}{|PR|} \qquad\qquad v = \frac{QP}{|QP|} + \frac{QR}{|QR|}$$

The incentre I is the intersection of the angle bisector passing through P, whose direction vector is u, and the one passing through Q, with direction vector v:

$$I = P + k\,u = Q + m\,v \qquad\qquad k,\ m \text{ real}$$

Arranging terms we find PQ as a linear combination of u and v:

$$k\,u - m\,v = Q - P = PQ$$

The coefficient k is:

$$k = \frac{PQ \wedge v}{u \wedge v} = \frac{|PQ|\,|RP|\,PQ \wedge QR}{PQ \wedge QR\,|RP| + QR \wedge RP\,|PQ| + RP \wedge PQ\,|QR|}$$

Since all outer products are equal because they are twice the triangle area, this expression is simplified:

$$k = \frac{|PQ|\,|RP|}{|RP| + |PQ| + |QR|}$$

Then, the centre of the circumscribed circle is:

$$I = P + k\,u = P + \frac{|PQ|\,|RP|}{|PQ|+|QR|+|RP|}\left(\frac{PQ}{|PQ|}+\frac{PR}{|PR|}\right)$$

By taking common denominator and simplifying, we arrive at:

$$I = \frac{P|QR|+Q|RP|+R|PQ|}{|QR|+|RP|+|PQ|}$$

For example, let us calculate the centre of the circle inscribed in the triangle with vertices:

$P(0, 0)$ $\quad Q(0, 3)$ $\quad R(4, 0)$

$|PQ| = 3$ $\quad |QR| = 5$ $\quad |RP| = 4$

$$I = \frac{5(0,0)+4(0,3)+3(4,0)}{5+4+3} = \frac{(12,12)}{12} = (1,1)$$

In order to find the radius, first we must obtain the segment IP:

$$IP = \frac{QP|RP|+RP|PQ|}{|QR|+|RP|+|PQ|}$$

The radius of the inscribed circle is the distance from I to side PQ:

$$d(I,PQ) = \frac{|IP\wedge PQ|}{|PQ|} = \frac{|RP\wedge PQ|}{|PQ|+|QR|+|RP|}$$

whence the ratio of radius follows:

$$\frac{\text{radius of circumscribed circle}}{\text{radius of inscribed circle}} = \frac{1}{2}\,\frac{|PQ|\,|QR|\,|RP|}{|PQ|+|QR|+|RP|}$$

Altitudes and orthocentre

The *altitude* of a side is the segment perpendicular to this side (also called *base*) that passes through the opposite vertex. The three altitudes of a triangle intersect at a unique point called *orthocentre*. Let us prove this statement by calculating the intersection H of two altitudes. Since H belongs to the altitude that is perpendicular to the base QR and passes through vertex P (figure 8.5), its equation is:

$$G = \frac{H + 2\ O}{3}$$

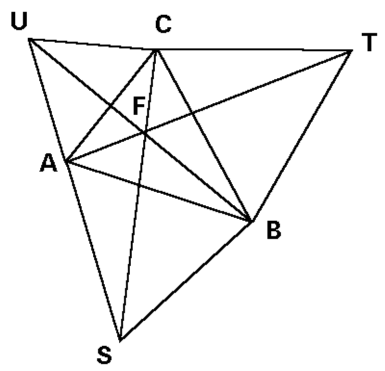
Hence the centroid is located between the orthocentre and the circumcentre, and its distance from the orthocentre is double its distance from the circumcentre.

Fermat's theorem

The geometric algebra allows us to prove Fermat's theorem in a very easy and intuitive way.

Over each side of a triangle ΔABC we draw an equilateral triangle (figure 8.7). Let T, U and S be the vertices of the equilateral triangles that are respectively opposite to A, B and C. Then, segments AT, BU and CS have the same length, form angles of $2\pi/3$ and intersect at a unique point F, called the *Fermat point*. Moreover, the addition of the three distances from any point P to each vertex is minimal when P is the Fermat point, provided that no interior angle of ΔABC is greater than $2\pi/3$.

Figure 8.7



First we must demonstrate that BU is obtained from AT by means of a rotation through $2\pi/3$, which will be represented by the complex number t:

$$AT\ t = (AC + CT\ )\ t = AC\ t + CT\ t \qquad\qquad t = \cos\frac{2\pi}{3} + e_{12}\sin\frac{2\pi}{3}$$

By construction, the vector AC turned through $2\pi/3$ is the vector CU, and CT turned through $2\pi/3$ is BC, so that:

$$AT\ t = CU + BC = BU$$

Analogously, one finds $CB = BU\ t$ and $AT = CS\ t$. Therefore, vectors CS, BU and AT have the same length and each of them is obtained from each other by successive rotations through $2\pi/3$.

Let us see that the sum of distances from P to the three vertices A, B and C is minimal when P is the Fermat point. First we must prove that the vector sum of PA turned through $4\pi/3$, PB turned through $2\pi/3$ and PC is constant and independent of the point P. (figure 8.8). That is, for any two points P and P' it is always true that:

$$PA\ t^2 + PB\ t + PC = P'A\ t^2 + P'B\ t + P'C$$

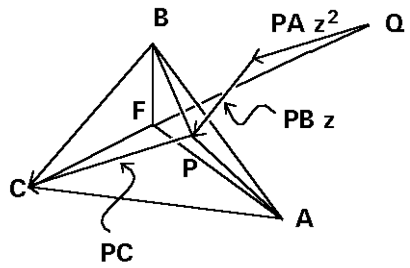
A fact that is easily proven by arranging all the terms on one side of the equation:

$$PP' (t^2 + t + 1) = 0$$

This product is always zero since $t^2 + t + 1 = 0$. Hence, there is a unique point Q such that:

$$PA\, t^2 + PB\, t + PC = QC$$

Figure 8.8



For any point P, the three segments form a broken line as shown in figure 8.8. Therefore, by the triangular inequality we have:

$$|PA| + |PB| + |PC| \geq |QC|$$

When P is the Fermat point F, these segments form a straight line. Then, the addition of the distances from F to the three vertices is minimal provided that no angle of the triangle is greater than $2\pi/3$:

$$|FA| + |FB| + |FC| = |QC| \leq |PA| + |PB| + |PC|$$

Otherwise, some vector among $FA\, t^2$, $FB\, t$, FC has a direction opposite to the others, so that its length is subtracted from the others and their sum is not minimal.

Exercises

8.1 Napoleon's theorem. Over each side of a generic triangle draw an equilateral triangle. Prove that the centres of these three equilateral triangles also form an equilateral triangle.

8.2 Leibniz's theorem. Let P be any point in the plane and G the centroid of a triangle ΔABC. Then $3\, PG^2 = PA^2 + PB^2 + PC^2 - (AB^2 + BC^2 + CA^2) / 3$.

8.3 Apollonius' lost theorem. Let A, B and C be three given points in the plane. Every point G in the plane can then be expressed as a linear combination of these three points (G is also considered as the centre of masses located at A, B and C with weights a, b and c [4]).

$$G = a\, A + b\, B + c\, C \qquad \text{with} \quad a + b + c = 1$$

Prove that:

a) a, b, c are the fractions of the area of ΔABC that are occupied by ΔGBC, ΔGCA and ΔGAB respectively.

b) The geometric locus of the points P in the plane such that $a\, PA^2 + b\, PB^2 + c\, PC^2 = k$ is a circle with centre G.

[4] See August Ferdinand Möbius, *Der Barycentrische Calcul* (1827), p. 17.

hyperbola, both distances are negative, so that the eccentricity is always positive. When e = 0 both foci are coincident at the centre of a circle. Note that a circle is obtained as the intersection of a cone with a horizontal plane. In this case, the directrices are the lines at infinity.

The *vectorial equation* of a conic is obtained from the polar equation and contains the radius vector *FP*. Since *FP* forms an angle β (figure 11.3) with *FQ*, *FP* is obtained from the unit vector of *FQ* via multiplication by the exponential of $\beta\, e_{12}$ and by the norm of *FP* yielding:

$$FP = \frac{1+\mathrm{e}}{1+\mathrm{e}\cos\beta}\, FQ\, (\cos\beta + e_{12}\sin\beta)$$

On the other hand, from the directrix property, one easily finds the following equation for a conic:

$$FP^2\, FT^2 = \mathrm{e}^2\, (\, FT^2 - FT\cdot FP\,)^2$$

F, *T* and e are parameters of the conic, and $P(x, y)$ is the mobile point. Therefore from this equation we will also obtain a Cartesian equation of second degree. For example, let us calculate the Cartesian equation of an ellipse with eccentricity ½, a focus at the point (3, 4) and a vertex at (4,5):

$$\mathrm{e} = 1/2 \qquad F = (\,3, 4\,) \qquad Q = (\,4, 5\,) \qquad P = (\,x, y\,)$$

$$FT = \frac{1+\mathrm{e}}{\mathrm{e}}\, FQ = 3\,e_1 + 3\,e_2 \qquad T = F + FT = (\,6, 7\,)$$

$$FP = (\,x-3\,)\,e_1 + (\,y-4\,)\,e_2$$

The equation of this conic is then:

$$[\,(\,x-3\,)^2 + (\,y-4\,)^2\,]\;18 = \frac{1}{4}\,[\,18 - (\,3\,(x-3) + 3\,(\,y-4\,)\,)\,]^2$$

and after simplification it is:

$$7\,x^2 - 2\,x\,y + 7\,y^2 - 22\,x - 38\,y + 31 = 0$$

Chasles' theorem

According to this theorem[3], the projective cross ratio of any four given points *A*, *B*, *C* and *D* on a conic regarded from a point *X* also lying on this conic is constant, independently of the choice of the point *X* (figure 11.7):

$$\{\,X, A\,B\,C\,D\,\} = \{\,X', A\,B\,C\,D\,\}$$

[3] Michel Chasles, *Traité des sections coniques*, Gauthier-Villars, Paris, 1865, p. 3.

To prove this theorem, let us take into account that points A, B, C, D, X must fulfil the vectorial equation of the conic. Let us also suppose, without loss of generality, the main axis of symmetry having the direction e_1 (this supposition simplifies calculations):

Figure 11.7

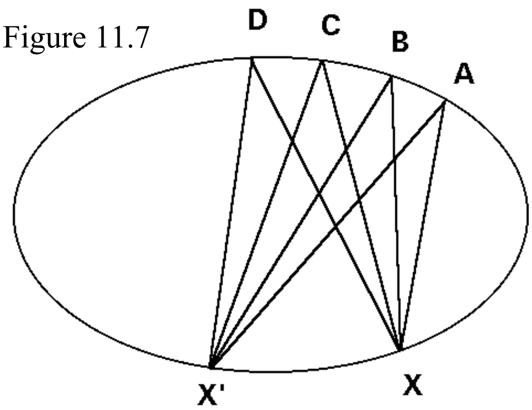
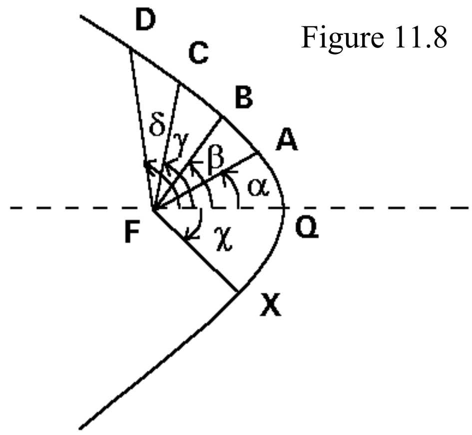


Figure 11.8



$$FQ = |FQ|\ e_1$$

From now on, α, β, γ, δ, χ will be the angles that the focal radii FA, FB, FC, FD, FX form with the main axis with direction vector FQ (figure 11.8). Then:

$$XA = FA - FX = |FQ|\,(1+\mathrm{e})\left[\frac{e_1\cos\alpha + e_2\sin\alpha}{1+\mathrm{e}\cos\alpha} - \frac{e_1\cos\chi + e_2\sin\chi}{1+\mathrm{e}\cos\chi}\right]$$

Introducing a common denominator, we find:

$$XA = |FQ|\frac{(1+\mathrm{e})\left[e_1(\cos\alpha - \cos\chi) + e_2(\sin\alpha - \sin\chi + \mathrm{e}\sin\alpha\cos\chi - \mathrm{e}\cos\alpha\sin\chi)\right]}{(1+\mathrm{e}\cos\alpha)(1+\mathrm{e}\cos\chi)}$$

From XA and the analogous expression for XC, and after simplification we obtain:

$$XA\wedge XC = FQ^2\,\frac{e_{12}\,(1+\mathrm{e})^2(\sin\gamma\cos\alpha - \sin\alpha\cos\gamma + \sin\chi\cos\gamma - \sin\gamma\cos\chi + \sin\alpha\cos\chi - \sin\chi\cos\alpha)}{(1+\mathrm{e}\cos\alpha)(1+\mathrm{e}\cos\gamma)(1+\mathrm{e}\cos\chi)}$$

$$= FQ^2\,\frac{(1+\mathrm{e})^2\left[\sin(\gamma-\alpha) + \sin(\chi-\gamma) + \sin(\alpha - \chi)\right]}{(1+\mathrm{e}\cos\alpha)(1+\mathrm{e}\cos\gamma)(1+\mathrm{e}\cos\chi)}\,e_{12}$$

Using the trigonometric the half-angle identities, the sum is converted into a product of sines (exercise 6.2):

$$XA\wedge XC = -4\,FQ^2\,\frac{(1+\mathrm{e})^2\left[\sin\left(\frac{\gamma-\alpha}{2}\right)\sin\left(\frac{\chi-\gamma}{2}\right)\sin\left(\frac{\alpha-\chi}{2}\right)\right]}{(1+\mathrm{e}\cos\alpha)(1+\mathrm{e}\cos\gamma)(1+\mathrm{e}\cos\chi)}\,e_{12}$$

Likewise, the other outer products are obtained. The projective cross ratio is their quotient, where the factors containing the eccentricity or the angle χ are simplified:

$$\{X, ABCD\} = \frac{XA \wedge XC \;\; XB \wedge XD}{XA \wedge XD \;\; XB \wedge XC} = \frac{\sin\dfrac{\gamma - \alpha}{2} \sin\dfrac{\delta - \beta}{2}}{\sin\dfrac{\delta - \alpha}{2} \sin\dfrac{\gamma - \beta}{2}} = \frac{\sin\dfrac{\angle AFC}{2} \sin\dfrac{\angle BFD}{2}}{\sin\dfrac{\angle AFD}{2} \sin\dfrac{\angle BFC}{2}}$$

since $\gamma - \alpha$ is the angle $\angle AFC$, etc. Therefore, the projective cross ratio of four points *A*, *B*, *C* and *D* on a conic is equal to the quotient of the sines of the focal half angles, which do not depend on *X*, but only on the positions of *A*, *B*, *C* and *D*, a fact that is the proof of Chasles' theorem. This statement is trivial for the case of a circle, because the inscribed angles are half the central angles. However, angles inscribed in a conic vary with the position of the point *X* and they differ from half focal angles. In spite of this, the fact that the quotient of the sines of inscribed angles (projective cross ratio) is equal to the quotient of the sines of half focal angles is a notable result. For the case of the hyperbola, remember that the focal radius of a point on the non-focal branch is oriented towards the focal branch but it has negative norm so that the focal angle is measured with respect to this orientation.

Tangent and perpendicular to a conic

The vectorial equation of a conic with the major diameter oriented in the direction e_1 (figure 11.9) is:

$$FP = \frac{(1+\mathrm{e})\,|FQ|}{1+\mathrm{e}\cos\alpha}\left(e_1 \cos\alpha + e_2 \sin\alpha\right)$$

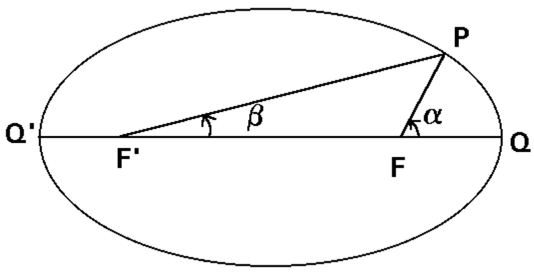


Figure 11.9

The derivation with respect to the angle α gives:

$$\frac{d\,FP}{d\alpha} = \frac{(1+\mathrm{e})\,|FQ|}{(1+\mathrm{e}\cos\alpha)^2}\left[-e_1 \sin\alpha + e_2\left(\mathrm{e} + \cos\alpha\right)\right]$$

This derivative has the direction of the line tangent to the conic at the point *P*, and its unit vector *t* is:

$$t = \frac{-e_1 \sin\alpha + e_2\left(\mathrm{e} + \cos\alpha\right)}{\sqrt{1 + \mathrm{e}^2 + 2\,\mathrm{e}\cos\alpha}}$$

The unit normal vector *n* is orthogonal to the tangent vector:

$$n = \frac{e_1\left(\mathrm{e} + \cos\alpha\right) + e_2 \sin\alpha}{\sqrt{1 + \mathrm{e}^2 + 2\,\mathrm{e}\cos\alpha}}$$

$$a + b\, e_1 = \begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix}$$

Now it is obvious that there should only be a unique root with odd index, which always exists for every hyperbolic number:

$$\sqrt[n]{a + b\, e_1} = \begin{pmatrix} \sqrt[n]{a+b} & 0 \\ 0 & \sqrt[n]{a-b} \end{pmatrix} \qquad n \text{ odd}$$

On the other hand, if $a + b > 0$ and $a - b > 0$ (right sector) there are four roots with even index, one in each sector:

n even

$$\begin{pmatrix} \sqrt[n]{a+b} & 0 \\ 0 & \sqrt[n]{a-b} \end{pmatrix} \in \text{right sector} \quad \begin{pmatrix} \sqrt[n]{a+b} & 0 \\ 0 & -\sqrt[n]{a-b} \end{pmatrix} \in \text{upper sector}$$

$$\begin{pmatrix} -\sqrt[n]{a+b} & 0 \\ 0 & -\sqrt[n]{a-b} \end{pmatrix} \in \text{left sector} \quad \begin{pmatrix} -\sqrt[n]{a+b} & 0 \\ 0 & \sqrt[n]{a-b} \end{pmatrix} \in \text{lower sector}$$

If the number does not belong to the right sector, some of the diagonal elements will be negative and there is no even root. This shows a panorama of the hyperbolic algebra far from that of complex numbers.

Hyperbolic analytic functions

What conditions should a hyperbolic function $f(z)$ of a hyperbolic variable z fulfil to be analytic? We want the derivative to be well defined:

$$\exists\, f^{I}(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

that is, this limit must be independent of the direction of Δz. If $f(z) = a + b\, e_1$ and the variable $z = x + y\, e_1$, the derivative calculated in the direction $\Delta z = \Delta x$ is then:

$$f^{I}(z) = \frac{\partial a}{\partial x} + e_1 \frac{\partial b}{\partial x}$$

while the derivative calculated in the direction $\Delta z = e_1\, \Delta y$ becomes:

$$f^{I}(z) = e_1 \frac{\partial a}{\partial y} + \frac{\partial b}{\partial y}$$

Both expressions must be equal, which results in the *conditions of hyperbolic analyticity*:

$$\frac{\partial a}{\partial x}=\frac{\partial b}{\partial y} \qquad \text{and} \qquad \frac{\partial a}{\partial y}=\frac{\partial b}{\partial x}$$

Note that exponential and logarithm fulfil these conditions and are therefore hyperbolic analytic functions. More exactly, the exponential function is analytic in the whole plane while the logarithm function is analytic in the left and right sectors, where the determinant of the hyperbolic numbers is positive.

By derivation of both identities one finds that analytic functions satisfy the following hyperbolic partial differential equation, called *one-dimensional wave equation*:

$$\frac{\partial^2 a}{\partial x^2}-\frac{\partial^2 a}{\partial y^2}=\frac{\partial^2 b}{\partial x^2}-\frac{\partial^2 b}{\partial y^2}=0$$

Now we must state the main integral theorem for hyperbolic analytic functions: <u>if a hyperbolic function is analytic in a certain region in the hyperbolic plane, then its integral following a closed path C within this region is zero</u>. If the hyperbolic function is $f(z)=a+b\,e_1$ then the integral is:

$$\oint_C f(z)\,dz=\oint_C (a+be_1)(dx+dy\,e_1)=\oint_C (a\,dx+b\,dy)+e_1\oint_C (a\,dy+b\,dx)$$

Since C is a closed path, we may apply Green's theorem to write:

$$=\iint_D\left(\frac{\partial b}{\partial x}-\frac{\partial a}{\partial y}\right)dx\wedge dy+e_1\iint_D\left(\frac{\partial a}{\partial x}-\frac{\partial b}{\partial y}\right)dx\wedge dy=0$$

where D is the region bounded by the closed path C. Since $f(z)$ fulfils the analyticity conditions everywhere within D, the integral vanishes.

From here, other theorems analogous to those of complex analysis follow, e. g.: <u>if $f(z)$ is a hyperbolic analytic function in a simply connected domain D and z_1 and z_2 are two points in D then the definite integral</u>:

$$\int_{z_2}^{z_1} f(z)\,dz$$

<u>between these points has a unique value independently of the integration path</u>.

Let us see an example. Consider the function $f(z) = 1 / (z - 1)$. This function is only defined if the inverse of $z - 1$ exists, which implies $|z - 1| \neq 0$. Of course, this function is analytic neither at $z = 1$ nor at the points:

$$|z-1|^2=0 \quad\Leftrightarrow\quad (x-1)^2-y^2=0 \quad\Leftrightarrow\quad (x+y-1)(x-y-1)=0$$

The lines $x + y = 1$ and $x - y = 1$ divide the analyticity domain into four sectors. Let us calculate the integral:

$$\int_{(5,-3)}^{(5,3)} \frac{dz}{z-1}$$

following two different paths in the right sector. The first one is a straight path given by the parametric equation $z = 5 + t\, e_1$ (figure 12.2):

$$\int_{(5,-3)}^{(5,3)} \frac{dz}{z-1} = \int_{-3}^{3} \frac{dt}{4+t\,e_1} e_1 = \int_{-3}^{3} \frac{\left(4-t\,e_1\right)dt}{16-t^2} e_1$$

Owing to symmetry, the integral of the odd function is zero:

$$= \int_{-3}^{3} \frac{4\,dt}{16-t^2} e_1 = \left[\frac{e_1}{2} \log\left|\frac{4+t}{4-t}\right| \right]_{-3}^{3} = e_1 \log 7$$

The second path (figure 12.2) is the hyperbola going from point (5, –3) to (5, 3):

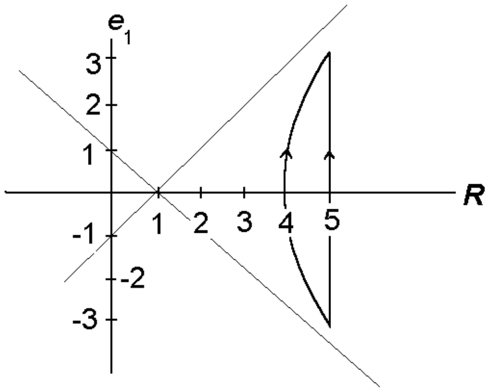


Figure 12.2

$$\int_{(5,-3)}^{(5,\,3)} \frac{dz}{z-1} = \int_{(5,-3)}^{(5,\,3)} \frac{\left(z^*-1\right)dz}{\left(z-1\right)\left(z^*-1\right)} = \int_{(5,-3)}^{(5,\,3)} \frac{\left(z^*-1\right)dz}{\det(z) - 2\,\mathrm{Re}(z) + 1}$$

Introducing its parametric equation, $z = 4\,(\cosh t + e_1 \sinh t)$ we have:

$$= \int_{-\log 2}^{\log 2} \frac{\left(4\cosh t - 4e_1 \sinh t - 1\right)\left(4\sinh t + 4\,e_1 \sinh t\right)}{16 - 8\cosh t + 1} dt = 4 \int_{-\log 2}^{\log 2} \frac{e_1\left(4-\cosh t\right) - 4\sinh t}{17 - 8\cosh t} dt$$

Due to symmetry, the integral of the hyperbolic sine (an odd function) divided by the denominator (an even function) is zero. We then split the integral into two integrals and find its value:

$$= \frac{15}{2} e_1 \int_{-\log 2}^{\log 2} \frac{dt}{17 - 8\cosh t} + e_1 \int_{-\log 2}^{\log 2} \frac{dt}{2} = \frac{e_1}{2} \left[\log \frac{-8\exp(t)+2}{-8\exp(t)+32} \right]_{-\log 2}^{\log 2} + e_1 \log 2 = e_1 \log 7$$

We now see that the integral following both paths gives the same result, as indicated by the theorem. In fact, the analytical functions can be integrated directly by using the indefinite integral:

$$\int_{(5,-3)}^{(5,3)} \frac{dz}{z-1} = \left[\log(z-1)\right]_{(5,-3)}^{(5,\,3)} = \left[\frac{1}{2} \log\left((x-1)^2 - y^2\right) + e_1 \operatorname{arg\,tanh}\left(\frac{y}{x-1}\right) \right]_{(5,-3)}^{(5,\,3)}$$

$$= \left[\frac{e_1}{2} \log\left|\frac{x-1+y}{x-1-y}\right| \right]_{(5,-3)}^{(5,\,3)} = e_1 \log 7$$

Consequently, the eccentricity e of the hyperbola is related to the obliquity ϕ of the transverse plane:

$$\cos\phi = \sqrt{e^2 - 1} \quad \text{with} \quad 1 < e < \sqrt{2}$$

The conjugate diameters of any hyperbola are intersections of this transverse plane with a pair of orthogonal axial planes; in other words, two radii are conjugate (figure 11.17) if their projections onto the horizontal plane are the semimajor and semiminor axes of the prism turned through the same hyperbolic angle φ:

$$OQ' = OQ\cosh\varphi + OS\sinh\varphi$$

$$OS' = OQ\sinh\varphi + OS\cosh\varphi$$

Our Euclidean eyes see these horizontal projections as symmetric lines with respect to the quadrant bisector. However, they are actually orthogonal because:

$$OQ'^2 - OS'^2 = OQ^2 - OS^2$$

and, therefore, they can be taken as a new system of orthogonal coordinates. We can even draw a new picture with the new diameters on the Cartesian axis.

The central equation of the hyperbola using the rotated axes is:

$$OP = \pm\left(OQ'\cosh(\psi - \varphi) + OS'\sinh(\psi - \varphi)\right)$$

which shows that a hyperbolic rotation of the coordinate axes has been made with respect to the principal diameters of the hyperbola.

The law of sines and cosines

Since the norm of the area is identical in both the Euclidean and the hyperbolic planes, a parallelogram is divided by its diagonal into two triangles with equal area. This statement is somewhat subtle since the Euclidean congruence of triangles is not valid in the hyperbolic plane. We will return to this question later. Now we only need to know that the area of a hyperbolic triangle is half the outer product of any two sides.

Following the perimeter of a triangle, let *a*, *b*, and *c* be its sides respectively opposite to the angles α, β and γ. Then, the angles formed by the oriented sides and the angles α, β and γ are supplementary:

$$a \wedge b = b \wedge c = c \wedge a \quad \Rightarrow \quad -|a||b|\sinh\gamma = -|b||c|\sinh\alpha = -|c||a|\sinh\beta$$

$$\frac{|a|}{\sinh\alpha} = \frac{|b|}{\sinh\beta} = \frac{|c|}{\sinh\gamma}$$

which is the *law of sines*.

Since a triangle is a closed polygon, $a + b + c = 0$ and we have:

$$a^2 = (-b-c)^2 = b^2 + c^2 + 2\,b \cdot c \qquad \Rightarrow \qquad a^2 = b^2 + c^2 - 2\,|\,b\,|\,|\,c\,|\cosh\alpha$$

which is the *law of cosines*. And also:

$$b^2 = a^2 + c^2 - 2\,|\,a\,|\,|\,c\,|\cosh\beta$$

$$c^2 = a^2 + b^2 - 2\,|\,a\,|\,|\,b\,|\cosh\gamma$$

When applying both theorems, we must be careful with the sides having imaginary length and the signs of the angles and trigonometric functions.

As an application of the law of sines and cosines, consider the hyperbolic triangle having the vertices $A(5, 3)$, $B(1, 0)$, $C(10, 1)$, whose sides have real norm (figure 13.8):

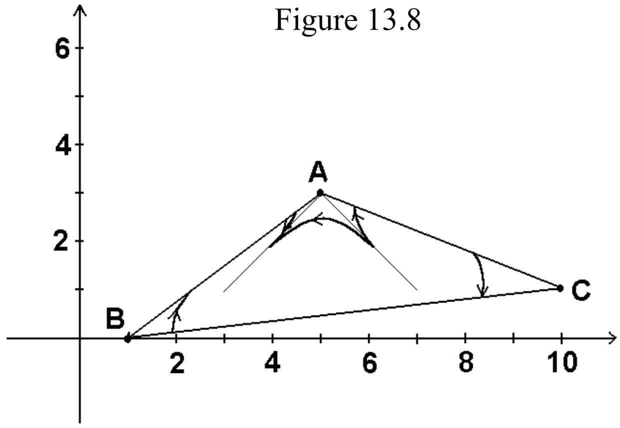


Figure 13.8

$$AB = B - A = -4\,e_2 - 3\,e_{21} \qquad |\,AB\,| = \sqrt{(-4)^2 - (-3)^2} = \sqrt{7}$$

$$BC = C - B = 9\,e_2 + e_{21} \qquad |\,BC\,| = \sqrt{9^2 - 1^2} = \sqrt{80}$$

$$CA = A - C = -5\,e_2 + 2\,e_{21} \qquad |\,CA\,| = \sqrt{(-5)^2 - 2^2} = \sqrt{21}$$

$$BC^2 = CA^2 + AB^2 - 2\,|\,CA\,|\,|\,AB\,|\cosh\alpha \qquad \cosh\alpha = -\frac{26}{7\sqrt{3}}$$

$$CA^2 = AB^2 + BC^2 - 2\,|\,AB\,|\,|\,BC\,|\cosh\beta \qquad \cosh\beta = \frac{33}{4\sqrt{35}}$$

$$AB^2 = BC^2 + CA^2 - 2\,|\,BC\,|\,|\,CA\,|\cosh\gamma \qquad \cosh\gamma = \frac{47}{4\sqrt{105}}$$

From where it follows that:

$$\alpha = -1.3966... - \pi\, e_{12} \qquad \beta = 0.8614... \qquad \gamma = 0.5352...$$

I have obtained their signs from the definition of the angles $\alpha = \angle BAC$, $\beta = \angle CBA$, $\gamma = \angle ACB$ and the geometric plot (figure 13.8). Note that $\alpha + \beta + \gamma = -\pi\, e_{12}$ and that they fulfil the law of sines:

$$\frac{|\,BC\,|}{\sinh\alpha} = \frac{|\,CA\,|}{\sinh\beta} = \frac{|\,AB\,|}{\sinh\gamma}$$

FOURTH PART: PLANE PROJECTIONS OF THREE-DIMENSIONAL SPACES

The complete study of the geometric algebra of the three-dimensional spaces falls out of the scope of this book. However, due to the importance of Earth charts and of Lobachevsky's geometry -the first one is more practical and the second one more theoretical-, I have written this last section. In order to make the explanations clearer, the three-dimensional geometric algebra has been reduced to the minimal concepts, enhancing the plane projections.

The geometric quality of being Euclidean or pseudo-Euclidean is not the signature + or − of a coordinate, but the fact that two coordinates have the same or different signature, in other words, it is a characteristic of the plane. For instance, a plane with signatures + + is equivalent, from a geometric point of view, to another with − − . Therefore, only two kinds of three-dimensional spaces exist: the room space where all the planes are Euclidean (signatures + + + or − − −), and the pseudo-Euclidean space, which has one Euclidean plane and two orthogonal pseudo-Euclidean planes (signatures + − − or + + −).

14. SPHERICAL GEOMETRY IN THE EUCLIDEAN SPACE

The geometric algebra of the Euclidean space

A *vector* of the Euclidean space is an oriented segment in this space with direction and sense that may represent other physical magnitudes such as forces, velocities, electric fields, etc. The set of all segments (geometric vectors) together with their addition (parallelogram rule) and their product by real numbers (dilation of vectors) has a structure of vector space, symbolised here by V_3. Every vector in V_3 has the form:

$$v = v_1\, e_1 + v_2\, e_2 + v_3\, e_3$$

where e_i are three unit perpendicular vectors, which are a basis of the space. If we define an associative product (*geometric* or *Clifford product*) as a generalisation of the multiplication of vectors in the Euclidean plane, we will arrive at:

$$e_i^2 = 1 \qquad \text{and} \qquad e_i\, e_j = -e_j\, e_i \ \text{ for } i \neq j$$

In general, the square of a vector is the square of its norm and perpendicular vectors anticommute whereas proportional vectors commute.

The geometric algebra generated by the space V_3 has eight dimensions:

$$Cl(V_3) = Cl_{3,\,0} = \langle 1, e_1, e_2, e_3, e_{23}, e_{31}, e_{12}, e_{123} \rangle$$

Let us see the product of two vectors in more detail:

$$v\, w = (v_1\, e_1 + v_2\, e_2 + v_3\, e_3)(w_1\, e_1 + w_2\, e_2 + w_3\, e_3) = v_1\, w_1 + v_2\, w_2 + v_3\, w_3$$
$$+ (v_2\, w_3 - v_3\, w_2)\, e_{23} + (v_3\, w_1 - v_1\, w_3)\, e_{31} + (v_1\, w_2 - v_2\, w_1)\, e_{12}$$

The product (or quotient) of two vectors is called a *quaternion*[1]. Quaternions are the even subalgebra of $Cl_{3,\,0}$ that generalise complex numbers to the three-dimensional space. Splitting a quaternion into the real and bivector parts, we obtain the *inner* (or *scalar*) product and the *outer* (or *exterior*) product respectively:

$$v \cdot w = v_1\, w_1 + v_2\, w_2 + v_3\, w_3$$

$$v \wedge w = \left(v_2\, w_3 - v_3\, w_2\right) e_{23} + \left(v_3\, w_1 - v_1\, w_3\right) e_{31} + \left(v_1\, w_2 - v_2\, w_1\right) e_{12}$$

Bivectors are oriented plane surfaces and indicate the direction of planes in space. Those who are acquainted with vector analysis will say that both vectors and bivectors are the same thing. This confusion was originated by Hamilton[2] himself, and continued by the founders of vector analysis, Gibbs and Heaviside. However, vectors and bivectors are different things just as physicists have experienced and know since a long time ago. The proper vectors are usually called "polar vectors" while the pseudo-vectors, which are actually bivectors, are usually called "axial vectors". The following magnitudes are vectors: of course a geometric segment, but also a velocity, an electric field, the momentum, etc. On the other hand, the oriented area is, of course, a bivector, but also the angular momentum, the angular velocity and the magnetic field. A criterion to distinguish both kinds of magnitudes is the reversal of coordinates, which changes the sense of vectors while leaves bivectors unchanged.

The product of two bivectors yields a real number plus a bivector. Both parts can be separated as symmetric and antisymmetric product. The symmetric product is a real number and its negative value will be denoted here by the symbol $\bullet$ while the antisymmetric product is also a bivector and will be denoted here by the symbol $\times$:

$$v \bullet w = -\frac{1}{2}\left(v\, w + w\, v\right) = v_{23}\, w_{23} + v_{31}\, w_{31} + v_{12}\, w_{12}$$

$$v \times w = -\frac{1}{2}\left(v\, w - w\, v\right)$$

$$= \left(v_{31}\, w_{12} - v_{12}\, w_{31}\right) e_{23} + \left(v_{12}\, w_{23} - v_{23}\, w_{12}\right) e_{31} + \left(v_{23}\, w_{31} - v_{31}\, w_{23}\right) e_{12}$$

$$v\, w = -v \bullet w - v \times w$$

Let us see what happens with the outer product of three vectors. According to the extension theory of Grassmann, the product $u \wedge v \wedge w$ is the oriented volume obtained from the surface the bivector $u \wedge v$ represents by a parallel translation along the segment w:

---

[1] From this definition, Hamilton deduced the multiplication rule of *i*, *j*, *k*. I recommend the reading of the initial chapters of the *Elements of Quaternions*, especially the section 2 "First Motive for naming the Quotient of two Vectors a Quaternion" in chapter I, p. 110.

[2] This confusion is due to the fact that vectors and bivectors are dual spaces of the algebra $Cl_{3,\,0}$. However, duality between vectors and bivectors does not exist at higher dimensions, although there are also dualities among other spaces.

15. HYPERBOLOIDAL GEOMETRY IN THE PSEUDO-EUCLIDEAN SPACE

The geometric algebra of the pseudo-Euclidean space

A *vector* in the pseudo-Euclidean space is an oriented segment in this space with direction and sense. The set of all segments (vectors) together with their addition (parallelogram rule) and the product by real numbers (dilation of vectors) has a structure of vector space, symbolised here by W_3. Every vector in W_3 has the form:

$$v = v_1\, e_1 + v_2\, e_2 + v_3\, e_3$$

where e_i are three unit perpendicular vectors, which constitute the basis of the space. The square of the norm of a vector is:

$$|\, v\,|^2 = v_1^2 + v_2^2 - v_3^2$$

which determines the geometric properties of this space, very different from the Euclidean space. Now we define an associative product (*geometric* or *Clifford product*) as a generalisation of those multiplications of vectors defined for the Euclidean and hyperbolic planes. Imposing the condition that the square of the norm must be equal to the square of the vector, we find:

$$|\, v\,|^2 = v^2$$

$$e_1^2 = e_2^2 = 1 \qquad e_3^2 = -1 \qquad \text{and} \qquad e_i\, e_j = -e_j\, e_i \quad \text{for } i \neq j$$

From the basis vectors one deduces that the geometric algebra generated by the space W_3 has eight components:

$$Cl(W_3) = Cl_{2,1} = \langle\ 1, e_1, e_2, e_3, e_{23}, e_{31}, e_{12}, e_{123}\ \rangle$$

Let us see with more detail the product of two vectors:

$$v\, w = (v_1\, e_1 + v_2\, e_2 + v_3\, e_3)(w_1\, e_1 + w_2\, e_2 + w_3\, e_3) = v_1\, w_1 + v_2\, w_2 - v_3\, w_3$$
$$+ (v_2\, w_3 - v_3\, w_2)\, e_{23} + (v_3\, w_1 - v_1\, w_3)\, e_{31} + (v_1\, w_2 - v_2\, w_1)\, e_{12}$$

I shall call the product (or quotient) of two vectors in W_3 a *tetranion*. Tetranions are the even subalgebra of $Cl_{2,1}$, which generalises complex and hyperbolic numbers to the pseudo-Euclidean space. Let t^* be the *conjugate* of a tetranion t.

$$t = a + b\, e_{23} + c\, e_{31} + d\, e_{12} \qquad\qquad t^* = a - b\, e_{23} - c\, e_{31} - d\, e_{12}$$

Then, the norm of t is given by:

$$|\, t\,| = \sqrt{t\, t^*} = \sqrt{a^2 - b^2 - c^2 + d^2}$$

because $e_{23}^2 = e_{31}^2 = 1$ and $e_{12}^2 = -1$. Consequently, $|e_{12}| = 1$, according to the fact that it represents a Euclidean plane whereas e_{23}, e_{31} have imaginary norm.

Splitting the tetranion product of two vectors into the real and bivector parts, we obtain the *inner* (or *scalar*) product and the *outer* (or *exterior*) product respectively:

$$v \cdot w = v_1\ w_1 + v_2\ w_2 - v_3\ w_3$$

$$v \wedge w = (v_2\ w_3 - v_3\ w_2)\, e_{23} + (v_3\ w_1 - v_1\ w_3)\, e_{31} + (v_1\ w_2 - v_2\ w_1)\, e_{12}$$

Here bivectors are also oriented plane surfaces indicating the direction of planes in the pseudo-Euclidean space. As before, vectors and bivectors are different things. Physics have also experienced this fact: in Minkowski's space, the electromagnetic field is a bivector whereas the tetrapotential is a vector. On the other hand, oriented areas are, of course, bivectors. As a criterion to distinguish both kinds of magnitudes one also uses the reversal of coordinates, which changes the sense of vectors while leaving invariant bivectors.

Two vectors are said to be *orthogonal* if their inner product vanishes:

$$v \perp w \quad \Leftrightarrow \quad v \cdot w = 0$$

Thus, the outer product is the product by the orthogonal component and the inner product is the product by the proportional component:

$$v \cdot w = v\ w_{\parallel} \qquad\qquad v \wedge w = v\ w_{\perp}$$

The product of two bivectors yields a tetranion. The real and bivector parts can be separated as symmetric and antisymmetric products. The symmetric product is a real number whose negative value will be denoted here by the symbol •, whereas the antisymmetric product with negative sign, denoted here by the symbol ×, is also a bivector:

$$v \bullet w = -\frac{1}{2}(v\ w + w\ v) = -v_{23}\ w_{23} - v_{31}\ w_{31} + v_{12}\ w_{12}$$

$$v \times w = -\frac{1}{2}(v\ w - w\ v)$$

$$= (v_{31}\ w_{12} - v_{12}\ w_{31})\, e_{23} + (v_{12}\ w_{23} - v_{23}\ w_{12})\, e_{31} - (v_{23}\ w_{31} - v_{31}\ w_{23})\, e_{12}$$

$$v\ w = -v \bullet w - v \times w$$

The outer product of three vectors has the same expression as for Euclidean geometry and this is a natural outcome of the extension theory: the product $u \wedge v \wedge w$ is the oriented volume generated by the surface represented by the bivector $u \wedge v$ when it is translated parallelly along the segment w:

$$u \wedge v \wedge w = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} e_{123}$$

Since $|e_{123}| = |e_{12}|\,|e_3|$, the pseudoscalar e_{123} has imaginary norm.

Finally, let us see how the product of three vectors *u*, *v* and *w* is. The vector *v* can be resolved into a component coplanar with *u* and *v* and another component perpendicular to the plane *u-v*:

$$u\, v\, w = u\, v_{\parallel}\, w + u\, v_{\perp}\, w$$

Let us analyse the permutative property now. For both Euclidean and hyperbolic planes we found $u\ v\ w - w\ v\ u = 0$. For the pseudo-Euclidean space the permutative property becomes:

$$u\, v\, w - w\, v\, u = u\, v_{\perp} w - w\, v_{\perp} u = v_{\perp}\left(-u\, w + w\, u\right)$$

$$= -2\, v_{\perp}\, u \wedge w = -2\, v \wedge u \wedge w = 2\, u \wedge v \wedge w$$

I take the same algebraic priorities as in the former chapter: all the abridged products must be operated before the geometric product, which is a convention adequate to the fact that the abridged products have to be developed as sums of geometric products.

The hyperboloid of two sheets (Lobachevskian surface)

According to Hilbert (*Grundlagen der Geometrie*, Anhang V) the complete Lobachevsky's "plane" cannot be represented by a smooth surface with a constant curvature as proposed by Beltrami. However, this result only concerns surfaces in the Euclidean space. The surface whose points are placed at a fixed distance from the origin in a pseudo-Euclidean space (the two-sheeted hyperboloid) is the surface sought by Hilbert that realises Lobachevsky's geometry[1]. It is known that it has a characteristic distance like the radius of the sphere. Since all the spheres are similar, we only needed to study the unit sphere. Likewise, all the hyperboloidal surfaces $x^2 + y^2 - z^2 = r^2$ are similar and the hyperboloid with imaginary unit radius (figure 15.1):

$$z^2 - x^2 - y^2 = 1$$

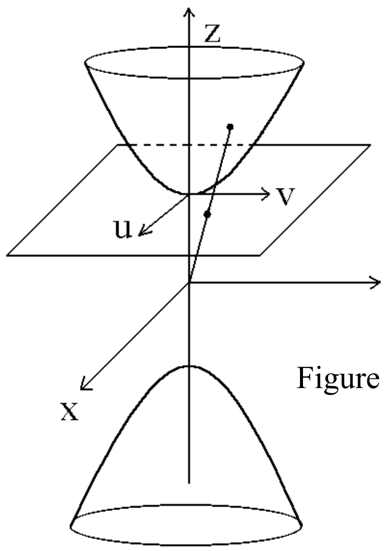


Figure 15.1

[1] The reader will find a complete study of the hyperboloidal surface in Faber, *Foundations of Euclidean and Non-Euclidean Geometry*, chap. VII. "The Weierstrass model".

$$\rho = \tanh \psi_0 \left(\frac{\tanh \frac{\psi}{2}}{\tanh \frac{\psi_0}{2}} \right)^n$$

About the congruence of geodesic triangles

Two geodesic triangles on the hyperboloid having the same angles also have the same sides and are said to be *congruent*. This follows immediately from the rotations in the pseudo-Euclidean space, which are expressed by means of tetranions like quaternions are used for the rotations of Euclidean space. However, this falls out of the scope of this book and will not be treated. The reader should perceive, in spite of his Euclidean eyes[9], that all the points on the hyperboloid are equivalent because the curvature is always the imaginary unit and the surface is always perpendicular to the radius. Thus the pole (vertex of the hyperboloid) is no special point, and any other point may be chosen as a new pole provided that the new axis are obtained from the old ones under a hyperbolic rotation.

The hyperboloid of one sheet

Whereas the Lobachevskian surface has imaginary unit radius, the one-sheeted hyperboloid (figure 15.9) has real unit radius and its equation is:

$$x^2 + y^2 - z^2 = 1$$

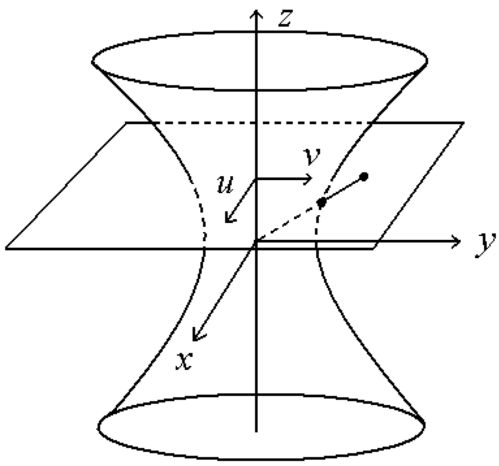


Figure 15.9

Under the central projection, the upper part of the hyperboloid is projected outside the Beltrami disk. However, we must take into account the fact that, if we want to represent the whole surface, each point in the projection plane has duplicity and represents two points: one on the upper part and the other on the lower part of the hyperboloid. This fact has significance and must be taken into account because the one-sheeted hyperboloid is a continuous surface. Something similar happens for the two-sheeted hyperboloid, but both sheets are not connected so that the Lobachevskian surface is only one of the sheets and projections apply to it.

Points lying in the *x*-*y* plane, which separate the upper and lower parts of the one-sheeted hyperboloid, are projected onto the line at infinity in the plane $z = 1$ while

[9] In fact, eyes are not Euclidean but a very perfect camera where images are projected onto a spherical surface. The principles of projection are likewise applicable to the pseudo-Euclidean space. Our mind, accustomed to the ordinary space (we learn its Euclidean properties in the first years of our life), deceives us when we wish to perceive the pseudo-Euclidean space.

points at infinity on the hyperboloid are projected (as for Lobachevsky's surface) onto the limit circle of the Beltrami disk. Geodesics are intersections of the hyperboloid with central planes[10] $z = a\,x + b\,y$. There are three kinds of geodesics depending on the orientation of the plane: hyperbolas if $a^2 + b^2>1$, ellipses if $a^2 + b^2<1$, straight lines and at the same time generatrices of the hyperboloid if $a^2 + b^2=1$. The central projection maps every geodesic onto a line that either cuts the limit circle (hyperbola) or does not intersect it (ellipse), or is tangent to the limit circle (generatrix of the hyperboloid).

The three former cases yield three different kinds of arc length: For hyperbolas the square of the arc length is negative; generatrices have null arc length; ellipses have real arc length such as Lobachevskian metric. Thus, the tangent plane to any point on the one-sheeted hyperboloid is a hyperbolic plane, in contrast with the fact that the tangent plane to any point on the Lobachevsky's surface is Euclidean.

Central projection and arc length on the one-sheeted hyperboloid

Since geodesics are projected onto straight lines under the central projection, it is the suitable projection to calculate arc lengths. The deduction steps resemble those for the two-sheeted hyperboloid although some signs change. As before:

$$u = \frac{x}{z} \qquad v = \frac{y}{z}$$

From the hyperboloid equation $x^2 + y^2 - z^2 = 1$ we obtain:

$$x = \frac{u}{\sqrt{u^2 + v^2 - 1}} \qquad y = \frac{v}{\sqrt{u^2 + v^2 - 1}} \qquad z = \frac{1}{\sqrt{u^2 + v^2 - 1}}$$

The differential of the arc length on the hyperboloid is:

$$ds = dx\, e_1 + dy\, e_2 + dz\, e_3$$

$$ds^2 = dx^2 + dy^2 - dz^2 = -\frac{\left(1 - v^2\right)du^2 + 2\,u\,v\,du\,dv + \left(1 - u^2\right)dv^2}{\left(u^2 + v^2 - 1\right)^2}$$

where the only difference with regard to Lobachevskian metric is a minus sign.

Geodesics are intersections of central planes with the hyperboloid and are projected onto straight lines under the central projection. So we take:

$$v = k\,u + l$$

Substitution into the arc length differential gives:

[10] This proof has been left as exercise 15.7.

12.4 From the decomposition theorem we have:

$$\sin(x+y\,e_1)=\frac{1+e_1}{2}\sin(x+y)+\frac{1-e_1}{2}\sin(x-y)=\sin x\cos y+e_1\cos x\sin y$$

12.5 From the analogous of de Moivre's identity we have:

$$(\cosh\psi+e_1\sinh\psi)^4\equiv\cosh 4\psi+e_1\sinh 4\psi$$

$$\cosh 4\psi\equiv\cosh^4\psi+6\cosh^2\psi\sinh^2\psi+\sinh^4\psi$$

$$\sinh 4\psi\equiv 4\cosh^3\psi\sinh\psi+4\cosh\psi\sinh^3\psi$$

12.6 The analytical continuation of the real logarithm is:

$$\log(x+y\,e_1)=\frac{1+e_1}{2}\log(x+y)+\frac{1-e_1}{2}\log(x-y)$$

$$=\frac{1}{2}\log(x^2-y^2)+\frac{e_1}{2}\log\frac{x+y}{x-y}$$

It may be rewritten in the form:

$$=\log\sqrt{x^2-y^2}+e_1\arg\tanh\frac{y}{x}$$

12.7 For the straight path $z=t\,e_1$ we have:

$$\int_{-e_1}^{e_1}z^2dz=e_1\int_{-1}^{1}t^2dt=e_1\left[\frac{t^3}{3}\right]_{-1}^{1}=\frac{2\,e_1}{3}$$

For the circular path $z=\cos t+e_1\sin t$ we have:

$$\int_{-e_1}^{e_1}z^2dz=\int_{-\pi/2}^{\pi/2}(\cos t+e_1\sin t)^2(-\sin t+e_1\cos t)\,dt$$

$$=\left[\cos t+\frac{2}{3}\cos^3 t+e_1\left(\sin t-\frac{2}{3}\sin^3 t\right)\right]_{-\pi/2}^{\pi/2}=\frac{2\,e_1}{3}$$

Using the indefinite integral, we find the same result:

$$\int_{-e_1}^{e_1}z^2dz=\left[\frac{z^3}{3}\right]_{-e_1}^{e_1}=\frac{2\,e_1}{3}$$

12.8 The proof is analogous to the exercise 3.12: turn the hyperbolic numbers *df*, *dz* into hyperbolic vectors by multiplying them by e_2 on the left.

12.9 Let us prove by induction that any power with positive exponent fulfils the decomposition theorem:

$$x + y\,e_1 = \frac{1+e_1}{2}(x+y) + \frac{1-e_1}{2}(x-y)$$

$$(x + y\,e_1)^{n+1} = (x + y\,e_1)^n\,(x + y\,e_1)$$

$$= \left[\frac{1+e_1}{2}(x+y)^n + \frac{1-e_1}{2}(x-y)^n\right]\left[\frac{1+e_1}{2}(x+y) + \frac{1-e_1}{2}(x-y)\right]$$

$$= \frac{1+e_1}{2}(x+y)^{n+1} + \frac{1-e_1}{2}(x-y)^{n+1} \qquad n \in N$$

We have already seen that:

$$\frac{1}{x + y\,e_1} = \frac{1+e_1}{2}\,\frac{1}{x+y} + \frac{1-e_1}{2}\,\frac{1}{x-y}$$

The fact that the powers with negative exponents also fulfil the decomposition theorem is also proven by induction in the same way as above:

$$\frac{1}{(x + y\,e_1)^n} = \frac{1+e_1}{2}\,\frac{1}{(x+y)^n} + \frac{1-e_1}{2}\,\frac{1}{(x-y)^n} \qquad n \in N$$

Therefore, the Laurent series also fulfils the decomposition theorem:

$$\sum_{n=-\infty}^{\infty} a_n\,(x + y\,e_1)^n = \frac{1+e_1}{2}\sum_{n=-\infty}^{\infty} a_n (x+y)^n + \frac{1-e_1}{2}\sum_{n=-\infty}^{\infty} a_n (x-y)^n \qquad n \in Z$$

and we have:

$$f(x + y\,e_1) = \frac{1+e_1}{2} f(x+y) + \frac{1-e_1}{2} f(x-y)$$

13. The hyperbolic or pseudo-Euclidean plane

13.1 If the vertices of the triangle are *A*(2, 2), *B*(1, 0) and *C*(5, 3) then:

AB = (–1, –2) *BC* = (4, 3) *CA* = (–3, –1)

CHRONOLOGY OF THE GEOMETRIC ALGEBRA

1679 Letters from Leibniz to Huygens on the *characteristica geometrica*.
1799 Publication of *Om Directionens analytiske Betegning* by Caspar Wessel with scarce diffusion.
1805 Birth of William Rowan Hamilton in Dublin.
1806 Publication of *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques* by Jean Robert Argand.
1809 Birth of Hermann Günther Grassmann in Stettin.
1818 Death of Wessel
1822 Death of Argand.
1827 Publication of *Der barycentrische Calcul* by Möbius in Leipzig.
1831 Birth of James Clerk Maxwell in Edinburgh.
1831 Birth of Peter Guthrie Tait.
1839 Birth of Josiah Willard Gibbs in New Haven.
1843 Discovery of the quaternions by Hamilton.
1844 Publication of the first edition of *Die Lineale Ausdehnungslehre* where Grassmann presents the anticommutative product of geometric unities (outer product).
1845 Birth of William Kingdon Clifford in Exeter.
1847 Publication of *Geometric Analysis* with a foreword by Möbius, memoir with which Grassmann won the prize that had been offered to whom could develop Leibniz's *characteristica geometrica*.
1850 Birth of Oliver Heaviside in London.
1853 Publication of *Lectures on Quaternions* where Hamilton introduces the nabla (gradient) operator.
1862 Publication of the second edition of *Die Ausdehnungslehre.*
1864 Publication of *A dynamical theory of the electromagnetic field* by Maxwell, where he defines the divergence and the curl.
1865 Death of Hamilton.
1866 Posthumous publication of Hamilton's *Elements of Quaternions*.
1867 Publication of *Elementary Treatise on Quaternions* by Tait.
1873 Publication of *Introduction to Quaternions* by Kelland and Tait.
1873 Maxwell publishes the *Treatise on Electricity and Magnetism* where he writes the equations of electromagnetic field with quaternions.
1877 Publication of Grassmann's paper "Der Ort der Hamilton'schen Quaternionen in der Ausdehnungslehre".
1877 Death of Grassmann.
1878 Publication of the paper "Applications of Grassmann's Extensive Algebra" by Clifford where he makes the synthesis of the systems of Grassmann and Hamilton.
1879 Death of Maxwell.
1879 Death of Clifford.
1880 Publication of Lipschitz's "Principes d'un calcul algébrique qui contient comme espèces particulières le calcul des quantités imaginaires et des quaternions".
1881 Private printing of *Elements of Vector Analysis* by Gibbs.
1886 Publication of Lipschitz' *Untersuchungen über die Summen von Quadraten*.
1886 Publication of Gibbs' paper "On multiple algebra".
1888 Publication of Peano's *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann preceduto dalle operazione della logica deduttiva*.

1891 Oliver Heaviside publishes “The elements of vectorial algebra and analysis” in *The Electrician Series*.
1895 Publication of Peano’s “Saggio di Calcolo Geometrico”.
1901 Death of Tait.
1901 Wilson publishes Gibbs’ lessons in *Vector Analysis*.
1903 Death of Gibbs.
1925 Death of Heaviside.
1926 Wolfgang Pauli introduces his matrices to explain the electronic spin.
1928 Publication of the paper “The Quantum Theory of Electron”, where Paul A. M. Dirac defines a set of 4×4 anticommutative matrices built from Pauli’s matrices.

This comparative diagram of life and works of the authors of (or related with) the geometric algebra visualises and summarises the chronology. The XIX century may be properly called *the century of the geometric algebra*. Note the premature death of Clifford, which caused the delay in the development of the geometric algebra throughout the XX century.

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               1800      1820      1840      1860      1880      1900      1920
                 |         |         |         |         |         |         |

Wessel*********************
Argand***********************
Hamilton            *******************************
Grassmann             ***********************************
Maxwell                          *************************
Tait                             ************************************
Gibbs                                *********************************
Clifford                                **ns****************
Heaviside                                 **************************************


Om Directionens   |
Essai sur ...       |
Der barycentrische Calcul      |
Die Ausdehnungslehre (1st ed.)         |
Lectures on Quaternions                     |
Die Ausdehnungslehre (2nd ed.)                  |
Elements of Quaternions                           |
Elementary Treatise on Quaternions                 |
Treatise on Electricity and Magnetism                 |
Der Ort der Hamilton'schen Quat.in der Ausdehnungslehre |
Applications of Grassmann's Extensive Algebra           |
Elements of Vector Analysis                               |
The Elements of Vectorial Algebra and Analysis                 |
Vector Analysis                                                     |
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